Solutions to PS #3.

2.12 Let $\{\Omega_\alpha\}$ be an open cover of the set $E := \{0\} \cup \{1/n \,|\, n \in \mathbb{N}\}$. Then for some α_0, $0 \in \Omega_{\alpha_0}$. Since Ω_{α_0} is open, $\exists r > 0$ s.t. $(-r, r) \subset \Omega_{\alpha_0}$. Since $1/n \to 0$, $\exists N \in \mathbb{N}$ s.t. $n \geq N \Rightarrow 1/n < r$; that is, $E \setminus (-r, r)$ (and hence $E \setminus \Omega_{\alpha_0} \subset E \setminus (-r, r)$) is a finite set or empty. If the latter, then Ω_{α_0} is a finite subcover all by itself. If the former, then we know there are finitely many $\Omega_{\alpha_1}, \Omega_{\alpha_2}, \ldots, \Omega_{\alpha_n}$ covering $E \setminus \Omega_{\alpha_0}$, and

$$E \subset \Omega_{\alpha_0} \cup \Omega_{\alpha_1} \cup \cdots \cup \Omega_{\alpha_n}.$$

$\star$1. Let $\{\Omega_\alpha\}$ be an open cover of $[a, b]$, and let

$$E := \{x \in [a, b] \,|\, [a, x] \text{ is covered by finitely many } \Omega_\alpha\}.$$

E is bounded above by b, so $\beta := \sup E$ exists with $\beta \leq b$. Since a is in some Ω_{α_0} with Ω_{α_0} open, $\beta > a$. And, it should be clear that, $\forall x \in E,\ [a, x] \subset E$.

First, we claim that $[a, \beta) \in E$. If this were not so, then there would be some $\epsilon > 0$ such that $E \cap (\beta - \epsilon, \beta] = \varnothing$. But, then $\beta - \epsilon$ would be an upper bound of E, which is impossible since $\beta = \sup E$.

Next, we claim that $\beta \in E$. For, since $\beta \leq b$, $\exists$ some Ω_{α_1} and some $r > 0$, s.t. $(\beta - r, \beta + r) \subset \Omega_{\alpha_1}$. Clearly the collection made up of a finite subcover of $[a, \beta)$ along with Ω_{α_1} forms a finite subcover of $[a, \beta]$.

Finally, we claim that $\beta = b$. For, were this not so, then $[a, \beta + r) \in E$, which would show that β was not even an upper bound of E, let alone the sup.