Solutions to PS #2

⋆1.

(i) ⇒ (ii): Let \( r > 0 \). By definition, there is a sequence of distinct points \((a_n)\) in \( E \) such that \( x_n \to a \). So, \( \exists N \in \mathbb{N} \) s.t. \( d(x_n, a) < r \) whenever \( n \geq N \); that is \( x_n \in B(a, r) \cap E, \forall n \geq N \).

(ii) ⇒ (iii): Obvious.

(iii) ⇒ (i): We must construct a sequence \((x_n)\) of distinct points in \( E \) converging to \( a \). Let \( r_0 = 1 \), and choose \( x_0 \in (B(a, r_0) \setminus \{a\}) \cap E \). Next, choose \( r_1 = \min\{2^{-1}, d(a, x_1)\} \), and a corresponding \( x_1 \in (B(a, r_1) \setminus \{a\}) \cap E \). In similar fashion, having chosen \( x_0, \ldots, x_{n-1} \), we choose \( x_n \in (B(a, r_n) \setminus \{a\}) \cap E \), where \( r_n := \min\{2^{-n}, d(a, x_{n-1})\} \).

We claim that \( x_n \to a \). To see this, let \( \epsilon > 0 \). By construction, \( \exists N \in \mathbb{N} \) s.t. \( r_n \leq 2^{-n} < \epsilon \), \( \forall n \geq N \). For these \( n \), \( d(x_n, a) < \epsilon \).

⋆2. Let \( x \in (E')' \). We must show that \( x \in E' \). By supposition, \( \exists \) a sequence of distinct points \((x_n)\) in \( E' \) s.t. \( x_n \to x \). But, by Theorem M.7,

\[
\begin{align*}
\exists y_1 \in E \text{ s.t. } d(y_1, x_1) &< 1, \\
\exists y_2 \in E \setminus \{y_1\} \text{ s.t. } d(y_2, x_2) &< \frac{1}{2}, \\
&\vdots \\
\exists y_n \in E \setminus \{y_1, y_2, \ldots, y_{n-1}\} \text{ s.t. } d(y_n, x_n) &< \frac{1}{n}, \\
&\vdots 
\end{align*}
\]

By construction, these \((y_n)\) form a sequence of distinct points in \( E \), and

\[
d(y_n, x) \leq d(y_n, x_n) + d(x_n, x) \leq \frac{1}{n} + d(x_n, x) \to 0 \text{ as } n \to \infty.
\]

Or, another way to prove \( E' \) is closed:

Let \( x \in (E')' \), and let \( r > 0 \). By Thm. M.7, it suffices to show that \( B(x, r) \cap E \) is infinite. To that end, take a sequence \((x_n)\) in \( E' \) s.t. \( x_n \to x \). \( \exists n_0 \in \mathbb{N} \) s.t. \( x_{n_0} \in B(x, r) \). Since \( B(x, r) \) is open, \( \exists s > 0 \) s.t. \( B(x_{n_0}, s) \subset B(x, r) \). And, since \( x_{n_0} \in E' \), \( B(x_{n_0}, s) \cap E \) is infinite (and hence \( B(x, r) \cap E \) as well).

⋆3. ⇒: We suppose that every subsequence of \((x_n)\) converges with limit \( x \). Since \((x_n)\) is a subsequence of itself, the result is immediate.
\(\Leftarrow\): Suppose \((x_n)\) converges to a limit \(x\), and let \((x_{n_j})\) be a subsequence. Fix an \(\epsilon > 0\). By definition \(\exists N \in \mathbb{N}\) s.t. \(n \geq N \Rightarrow d(x_n, x) < \epsilon\). Take \(J \in \mathbb{N}\) sufficiently large (in fact, \(J = N\) will do) so that \(n_j \geq N\) whenever \(j \geq J\). Then \(d(x_{n_j}, x) < \epsilon\) for \(j \geq J\). Since \(\epsilon\) was arbitrarily chosen, \(x_{n_j} \rightarrow x\). And, this convergence to \(x\) is irrespective of the particular subsequence of \((x_n)\) chosen, so we have the result.

2.6 That \(E' \subset \overline{E}'\) is clear. We must prove the other inclusion. To do so, let \(x \in \overline{E}'\), and choose \(r > 0\). We will show that \(B(x, r) \cap E\) is an infinite set. To that end, let \((x_n)\) be a sequence of distinct points in \(\overline{E}\) with \(x_n \rightarrow x\). Choose \(N\) large enough so that \(d(x_N, x) < r/2\). Set \(\epsilon = d(x_N, x)\). Since \(x_N \in E'\), \((B(x_N, \epsilon) \cap E) \subset (B(x, r) \cap E)\) is infinite.

Since \(E'\) is closed, \((E')' \subset E'\), which shows that the limit points of \(E'\) are limits points of \(E\) as well. To see that this set inclusion need not hold in the other direction, consider the set \(E := \{1/n \mid n \in \mathbb{N}\} \subset \mathbb{R}\). With this \(E\) we have \(E' = \{0\}\) and \((E')' = \emptyset\).

2.7 (a) Let \(n \in \mathbb{N}\).

\(\subset\): Let \(x \in \overline{B}_n\). By definition, \(\exists\) a sequence \((x_j)\) in \(B_n\) s.t. \(x_j \rightarrow x\). If \(\{x_j \mid j \in \mathbb{N}\}\) is a finite set, then \(x_N = x\) for some (in fact, infinitely many) \(N \in \mathbb{N}\). Since \(x_N \in B_n\), \(x_N\) is in at least one of \(A_1, \ldots, A_n\), showing \(x\) to be in the set on the right-hand side.

So, let us suppose that \(\{x_j \mid j \in \mathbb{N}\}\) is an infinite set. By the pigeonhole principle, some (fixed) \(A_p\) must contain a subsequence \((x_{j_k})\) of \((x_j)\). By the first problem in this set, \(x_{j_k} \rightarrow x\). So, \(x \in \overline{A}_p\) \(\Rightarrow\) \(x \in \bigcup_{i=1}^n \overline{A}_i\).

\(\supset\): Let \(x \in \bigcup_{i=1}^n \overline{A}_i\). Then \(x \in \overline{A}_p\) for some integer \(1 \leq p \leq n\), and \(\exists\) a sequence \((x_j)\) in \(A_p \subset B_n\) s.t. \(x_j \rightarrow x\).

(b) The proof is about the same as the “\(\supset\)" inclusion above. Let \(x \in \bigcup_{i=1}^\infty \overline{A}_i\). Then \(x \in \overline{A}_p\) for some \(p\), and \(\exists\) a sequence \((x_j)\) in \(A_p \subset \bigcup_{i=1}^\infty A_i\) s.t. \(x_j \rightarrow x\). Thus, \(x \in \bigcup_{i=1}^\infty \overline{A}_i\).

Here are a couple counterexamples:

1. For each \(n \in \mathbb{N}\), take \(A_n = \{1/n\}\). Then

\[
\bigcup_{n=1}^\infty A_n = \bigcup_{n=1}^\infty \overline{A}_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}, \quad \text{but} \quad \bigcup_{n=1}^\infty A_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0\}.
\]
2. For each $n \in \mathbb{N}$, take $A_n = [1/n, 1)$. Then

$$\bigcup_{n=1}^{\infty} A_n = (0, 1),$$
$$\bigcup_{n=1}^{\infty} \overline{A_n} = (0, 1], \quad \text{and}$$
$$\bigcup_{n=1}^{\infty} A_n = [0, 1].$$