Solutions to PS #2

$\star$1.

(i) $\Rightarrow$ (ii): Let $r > 0$. By definition, there is a sequence of distinct points (a_n) in E such that $x_n \to a$. So, $\exists N \in \mathbb{N}$ s.t. $d(x_n, a) < r$ whenever $n \geq N$; that is $x_n \in B(a,r) \cap E$, $\forall n \geq N$.

(ii) $\Rightarrow$ (iii): Obvious.

(iii) $\Rightarrow$ (i): We must construct a sequence (x_n) of distinct points in E converging to a. Let $r_0 = 1$, and choose $x_0 \in (B(a,r_0) \setminus \{a\}) \cap E$. Next, choose $r_1 = \min\{2^{-1}, d(a,x_1)\}$, and a corresponding $x_1 \in (B(a,r_1) \setminus \{a\}) \cap E$. In similar fashion, having chosen $x_0, \ldots, x_{n-1}$, we choose $x_n \in (B(a,r_n) \setminus \{a\}) \cap E$, where $r_n := \min\{2^{-n}, d(a, x_{n-1})\}$.

We claim that $x_n \to a$. To see this, let $\epsilon > 0$. By construction, $\exists N \in \mathbb{N}$ s.t. $r_n \leq 2^{-n} < \epsilon$, $\forall n \geq N$. For these n, $d(x_n, a) < \epsilon$.

$\star$2. Let $x \in (E')'$. We must show that $x \in E'$. By supposition, $\exists$ a sequence of distinct points (x_n) in E' s.t. $x_n \to x$. But, by Theorem M.7,

$$\begin{aligned}
&\exists y_1 \in E \text{ s.t. } d(y_1, x_1) < 1, \\
&\exists y_2 \in E \setminus \{y_1\} \text{ s.t. } d(y_2, x_2) < \frac{1}{2}, \\
&\vdots \\
&\exists y_n \in E \setminus \{y_1, y_2, \ldots, y_{n-1}\} \text{ s.t. } d(y_n, x_n) < \frac{1}{n}, \\
&\vdots
\end{aligned}$$

By construction, these (y_n) form a sequence of distinct points in E, and

$$d(y_n, x) \;\leq\; d(y_n, x_n) + d(x_n, x) \;\leq\; \frac{1}{n} + d(x_n, x) \;\to\; 0 \text{ as } n \to \infty.$$

Or, another way to prove E' is closed:
Let $x \in (E')'$, and let $r > 0$. By Thm. M.7, it suffices to show that $B(x,r) \cap E$ is infinite. To that end, take a sequence (x_n) in E' s.t. $x_n \to x$. $\exists\, n_0 \in \mathbb{N}$ s.t. $x_{n_0} \in B(x,r)$. Since $B(x,r)$ is open, $\exists s > 0$ s.t. $B(x_{n_0}, s) \subset B(x,r)$. And, since $x_{n_0} \in E'$, $B(x_{n_0}, s) \cap E$ is infinite (and hence $B(x,r) \cap E$ as well).

$\star$3. $\Rightarrow$: We suppose that every subsequence of (x_n) converges with limit x. Since (x_n) is a subsequence of itself, the result is immediate.

$\Leftarrow$: Suppose (x_n) converges to a limit x, and let (x_{n_j}) be a subsequence. Fix an $\epsilon > 0$. By definition $\exists\, N \in \mathbb{N}$ s.t. $n \geq N \;\Rightarrow\; d(x_n, x) < \epsilon$. Take $J \in \mathbb{N}$ sufficiently large (in fact, $J = N$ will do) so that $n_j \geq N$ whenever $j \geq J$. Then $d(x_{n_j}, x) < \epsilon$ for $j \geq J$. Since ϵ was arbitrarily chosen, $x_{n_j} \to x$. And, this convergence to x is irrespective of the particular subsequence of (x_n) chosen, so we have the result.

2.6 That $E' \subset \overline{E}\,'$ is clear. We must prove the other inclusion. To do so, let $x \in \overline{E}\,'$, and choose $r > 0$. We will show that $B(x, r) \cap E$ is an infinite set. To that end, let (x_n) be a sequence of distinct points in $\overline{E}$ with $x_n \to x$. Choose N large enough so that $d(x_N, x) < r/2$. Set $\epsilon = d(x_N, x)$. Since $x_N \in E'$, $(B(x_N, \epsilon) \cap E) \subset (B(x, r) \cap E)$ is infinite.

Since E' is closed, $(E')\,' \subset E'$, which shows that the limit points of E' are limits points of E as well. To see that this set inclusion need not hold in the other direction, consider the set $E := \{1/n \,|\, n \in \mathbb{N}\} \subset \mathbb{R}$. With this E we have $E' = \{0\}$ and $(E')' = \varnothing$.

2.7 (a) Let $n \in \mathbb{N}$.

$\subset$: Let $x \in \overline{B}_n$. By definition, $\exists$ a sequence (x_j) in B_n s.t. $x_j \to x$. If $\{x_j \,|\, j \in \mathbb{N}\}$ is a finite set, then $x_N = x$ for some (in fact, infinitely many) $N \in \mathbb{N}$. Since $x_N \in B_n$, x_N is in at least one of $A_1, \ldots, A_n$, showing x to be in the set on the right-hand side.

So, let us suppose that $\{x_j \,|\, j \in \mathbb{N}\}$ is an infinite set. By the pigeonhole principle, some (fixed) A_p must contain a subsequence (x_{j_k}) of (x_j). By the first problem in this set, $x_{j_k} \to x$. So, $x \in \overline{A}_p \;\Rightarrow\; x \in \bigcup_{i=1}^{n} \overline{A}_i$.

$\supset$: Let $x \in \bigcup_{i=1}^{n} \overline{A}_i$. Then $x \in \overline{A}_p$ for some integer $1 \leq p \leq n$, and $\exists$ a sequence (x_j) in $A_p \subset B_n$ s.t. $x_j \to x$.

(b) The proof is about the same as the "$\supset$" inclusion above. Let $x \in \bigcup_{i=1}^{\infty} \overline{A}_i$. Then $x \in \overline{A}_p$ for some p, and $\exists$ a sequence (x_j) in $A_p \subset \bigcup_{i=1}^{\infty} A_i$ s.t. $x_j \to x$. Thus, $x \in \overline{\bigcup_{i=1}^{\infty} A_i}$.

Here are a couple counterexamples:

1. For each $n \in \mathbb{N}$, take $A_n = \{1/n\}$. Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n &= \bigcup_{n=1}^{\infty} \overline{A}_n \\ &= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}, \qquad \text{but} \\ \overline{\bigcup_{n=1}^{\infty} A_n} &= \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\} \cup \{0\}. \end{aligned}$$

2. For each $n \in \mathbb{N}$, take $A_n = [1/n, 1)$. Then

$$\begin{aligned} \bigcup_{n=1}^{\infty} A_n &= (0,1), \\ \bigcup_{n=1}^{\infty} \overline{A}_n &= (0,1], \qquad \textit{and} \\ \overline{\bigcup_{n=1}^{\infty} A_n} &= [0,1]. \end{aligned}$$

,