Solutions to PS #1.

⋆1. Let $x, y, z \in Y$. Since in $x, y, z \in X$ and $d$ is a metric on $X$, it is clear that all four properties hold:

(i) $d(x, x) = 0$,
(ii) $d(x, y) > 0$, if $x \neq y$,
(iii) $d(x, y) = d(y, x)$, and
(iv) $d(x, y) \leq d(x, z) + d(z, y)$.

Since these elements were chosen arbitrarily from $Y$, $d$ is a metric on $Y$, and hence $(Y, d)$ is a metric space.

⋆2. Since $(x_n)$ is Cauchy, $\exists N \in \mathbb{N}$ st $d(x_m, x_N) < 1$ whenever $m \geq N$. That is, the set of points

$$\{x_N, x_{N+1}, x_{N+2}, \ldots\} \subset B(x_N, 1).$$

Now let

$$r := \max_{1 \leq j \leq N-1} d(x_j, x_N),$$

and choose $R := \max\{2r, 1\}$. Then

$$\{x_j | j \in \mathbb{N}\} \subset B(x_N, R).$$

⋆3. Note that if we rephrase this result as

$$(x_n) \text{ converges} \iff (x_n) \text{ is Cauchy},$$

the “⇒” part corresponds to the “only if” statement.

⇐: This case is easy, as we assume both that $(x_n)$ is Cauchy, and that $(X, d)$ is complete (i.e., all Cauchy sequences are convergent).

⇒: We assume that $(x_n)$ converges, say, to a point $x \in X$. Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon/2$ whenever $n \geq N$. Then, for $m, n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
Naturally, if $X$ is not complete, the "$\Leftarrow$" assertion is not so straightforward to prove. Indeed, it is no longer true. As an example, consider the interval $(0,1]$ with the usual metric $d$ of $\mathbb{R}$. By the first problem, $((0,1], d)$ is a metric space. Nevertheless, the sequence
$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$
is Cauchy but does not converge to a point of $(0,1]$.

2.11 The function $d_3$ is not a metric on $\mathbb{R}^1$, since $d_3(-1,1) = 0$. Similarly, $d_4$ may be ruled out since $d_4(2,1) = 0$. On the other hand, it relatively easy to see that properties (i), (ii) and (iii) of Defn. 2.15 (how I numbered them in class) hold for $d_1$, $d_2$ and $d_5$. Hence, each must be checked to see if (iv) (the triangle inequality) holds. We check these individually below.

For $d_1$, condition (iv) requires that
$$(x - y)^2 \leq (x - z)^2 + (z - y)^2, \quad \forall x, y, z \in \mathbb{R}.$$Expanding and eliminating terms common to both sides, we see the following equivalences:
$$(x - y)^2 \leq (x - z)^2 + (z - y)^2 \quad \Leftrightarrow \quad z^2 + xy - xz - zy \geq 0 \quad \Leftrightarrow \quad (z - y)(z - x) \geq 0.$$The last of these clearly does not hold for all choices of $x, y, z \in \mathbb{R}$ (take $x = 1$, $y = -1$, $z = 0$ for instance), so the first does not hold for all real $x, y, z$ either.

For $d_2$, there are the equivalences
$$\sqrt{|x - y|} \leq \sqrt{|x - z|} + \sqrt{|z - y|} \quad \Leftrightarrow \quad |x - y| \leq |x - z| + 2\sqrt{|x - z||z - y|} + |z - y|.$$But the last of these holds for all $x, y, z \in \mathbb{R}$ because the usual metric on $\mathbb{R}^1$ satisfies the triangle inequality—that is,
$$|x - y| \leq |x - z| + |z - y|$$and $\sqrt{|x - z||z - y|} \geq 0$, so adding it to the right-hand side only makes that side bigger. Tracing back these equivalences, we see that
$$\sqrt{|x - y|} \leq \sqrt{|x - z|} + \sqrt{|z - y|}, \quad \forall x, y, z \in \mathbb{R},$$and hence $d_2$ is a metric on $\mathbb{R}$. 
Let \( u := |x - y|, v := |x - z| \) and \( w := |z - y| \). Then \( u, v, w \geq 0 \) and, by the triangle inequality (applied to the usual metric on \( \mathbb{R} \)), \( u \leq v + w \). What we must show, if \( d_5 \) is to be a metric, is that \( f(u) \leq f(v) + f(w) \), where \( f(t) := t/(1 + t) \). Notice that this appears likely to be the case, since the graph of \( f \) for \( t \geq 0 \) is increasing (\( f'(t) = (1 + t)^{-2} \)) and concave down (so we would expect values of \( f \) at two points \( v, w \in [0, \infty) \) to have a sum at least as large as \( f(u) \) when \( u \leq v + w \)). A way to prove this is with integrals. Since \( f(0) = 0 \), for \( t \geq 0 \) we have

\[
f(t) = \int_0^t \frac{ds}{(1 + s)^2}.
\]

Thus

\[
\begin{align*}
  f(v) + f(w) & = \int_0^v \frac{ds}{(1 + s)^2} + \int_0^w \frac{ds}{(1 + s)^2} \\
  & = \int_0^v \frac{ds}{(1 + s)^2} + \int_v^{v+w} \frac{d\zeta}{(1 + \zeta - v)^2} \quad \text{(substituting } \zeta = s + v) \\
  & = \int_0^v \frac{ds}{(1 + s)^2} + \int_v^{v+w} \frac{ds}{(1 + s - v)^2} \\
  & \geq \int_0^v \frac{ds}{(1 + s)^2} + \int_v^{v+w} \frac{ds}{(1 + s)^2} \quad \text{(bigger denominator, smaller integrand)} \\
  & = \int_0^{v+w} \frac{ds}{(1 + s)^2} \\
  & = f(v + w) \\
  & \geq f(u) \quad \text{(since } f \text{ is increasing)}.
\end{align*}
\]