Solutions to PS #1.

$\star$1. Let x, y, $z \in Y$. Since in x, y, $z \in X$ and d is a metric on X, it is clear that all four properties hold:

(i) $d(x,x) = 0$,

(ii) $d(x,y) > 0$, if $x \neq y$,

(iii) $d(x,y) = d(y,x)$, and

(iv) $d(x,y) \leq d(x,z) + d(z,y)$.

Since these elements were chosen arbitrarily from Y, d is a metric on Y, and hence (Y,d) is a metric space.

$\star$2. Since (x_n) is Cauchy, $\exists N \in \mathbb{N}$ *st* $d(x_m, x_N) < 1$ whenever $m \geq N$. That is, the set of points

$$\{x_N, x_{N+1}, x_{N+2}, \ldots\} \subset B(x_N, 1).$$

Now let

$$r := \max_{1 \leq j \leq N-1} d(x_j, x_N),$$

and choose $R := \max\{2r, 1\}$. Then

$$\{x_j \,|\, j \in \mathbb{N}\} \subset B(x_N, R).$$

$\star$3. Note that if we rephrase this result as

$$(x_n) \text{ converges} \qquad \Leftrightarrow \qquad (x_n) \text{ is Cauchy},$$

the "$\Rightarrow$" part corresponds to the "only if" statement.

$\Leftarrow$: This case is easy, as we assume both that (x_n) is Cauchy, and that (X,d) is complete (i.e., all Cauchy sequences are convergent).

$\Rightarrow$: We assume that (x_n) converges, say, to a point $x \in X$. Let $\epsilon > 0$, and choose $N \in \mathbb{N}$ s.t. $d(x_n, x) < \epsilon/2$ whenever $n \geq N$. Then, for m, $n \geq N$, we have

$$d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Naturally, if X is not complete, the "$\Leftarrow$" assertion is not so straightforward to prove. Indeed, it is no longer true. As an example, consider the interval $(0, 1]$ with the usual metric d of $\mathbb{R}$. By the first problem, $((0, 1], d)$ is a metric space. Nevertheless, the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

is Cauchy but does not converge to a point of $(0, 1]$.

2.11 The function d_3 is not a metric on $\mathbb{R}^1$, since $d_3(-1, 1) = 0$. Similarly, d_4 may be ruled out since $d_4(2, 1) = 0$. On the other hand, it relatively easy to see that properties (i), (ii) and (iii) of Defn. 2.15 (how I numbered them in class) hold for d_1, d_2 and d_5. Hence, each must be checked to see if (iv) (the triangle inequality) holds. We check these individually below.

For d_1, condition (iv) requires that

$$(x - y)^2 \;\le\; (x - z)^2 + (z - y)^2, \qquad \forall x, y, z \in \mathbb{R}.$$

Expanding and eliminating terms common to both sides, we see the following equivalences:

$$\begin{aligned} (x - y)^2 \;\le\; (x - z)^2 + (z - y)^2 \qquad &\Leftrightarrow \qquad z^2 + xy - xz - zy \;\ge\; 0 \\ &\Leftrightarrow \qquad (z - y)(z - x) \ge 0. \end{aligned}$$

The last of these clearly does not hold for all choices of x, y, $z \in \mathbb{R}$ (take $x = 1$, $y = -1$, $z = 0$ for instance), so the first does not hold for all real x, y, z either.

For d_2, there are the equivalences

$$\sqrt{|x - y|} \;\le\; \sqrt{|x - z|} + \sqrt{|z - y|} \qquad \Leftrightarrow \qquad |x - y| \;\le\; |x - z| + 2\sqrt{|x - z||z - y|} + |z - y|.$$

But the last of these holds for all x, y, $z \in \mathbb{R}$ because the usual metric on $\mathbb{R}^1$ satisfies the triangle inequality—that is,

$$|x - y| \;\le\; |x - z| + |z - y|$$

—and $\sqrt{|x - z||z - y|} \ge 0$, so adding it to the right-hand side only makes that side bigger. Tracing back these equivalences, we see that

$$\sqrt{|x - y|} \;\le\; \sqrt{|x - z|} + \sqrt{|z - y|}, \qquad \forall x, y, z \in \mathbb{R},$$

and hence d_2 is a metric on $\mathbb{R}$.

Let $u := |x - y|$, $v := |x - z|$ and $w := |z - y|$. Then u, v, $w \geq 0$ and, by the triangle inequality (applied to the usual metric on $\mathbb{R}$), $u \leq v + w$. What we must show, if d_5 is to be a metric, is that $f(u) \leq f(v) + f(w)$, where $f(t) := t/(1 + t)$. Notice that this appears likely to be the case, since the graph of f for $t \geq 0$ is increasing ($f'(t) = (1 + t)^{-2}$) and concave down (so we would expect values of f at two points v, $w \in [0, \infty)$ to have a sum at least as large as $f(u)$ when $u \leq v + w$). A way to prove this is with integrals. Since $f(0) = 0$, for $t \geq 0$ we have

$$f(t) \;=\; \int_0^t \frac{ds}{(1+s)^2}.$$

Thus

$$\begin{aligned}
f(v) + f(w) &= \int_0^v \frac{ds}{(1+s)^2} + \int_0^w \frac{ds}{(1+s)^2} \\
&= \int_0^v \frac{ds}{(1+s)^2} + \int_v^{v+w} \frac{d\zeta}{(1+\zeta-v)^2} \qquad \text{(substituting } \zeta = s + v\text{)} \\
&= \int_0^v \frac{ds}{(1+s)^2} + \int_v^{v+w} \frac{ds}{(1+s-v)^2} \\
&\geq \int_0^v \frac{ds}{(1+s)^2} + \int_v^{v+w} \frac{ds}{(1+s)^2} \qquad \text{(bigger denominator, smaller integrand)} \\
&= \int_0^{v+w} \frac{ds}{(1+s)^2} \\
&= f(v+w) \\
&\geq f(u) \qquad \text{(since } f \text{ is increasing).}
\end{aligned}$$