MATH 362: Problem Set #24

⋆38. Prove Proposition L.16: Let X be a nonempty set, $\mathcal{C} \subset \mathcal{P}(X)$. Then $\exists$ a smallest σ-algebra containing $\mathcal{C}$ (in the sense that all other σ-algebras containing $\mathcal{C}$ contain this one.

Hint: Consider the collection

$$\mathscr{A} := \{\mathcal{L} \,|\, \mathcal{L} \text{ is a } \sigma\text{-algebra with } \mathcal{C} \in \mathcal{L}\} \ .$$

Show that $\mathscr{A}$ is nonempty. Then let Σ be the intersection over all σ-algebras contained in $\mathscr{A}$. Show that Σ is a σ-algebra, that it contains $\mathcal{C}$, and that it is contained in every σ-algebra which contains $\mathcal{C}$.

⋆39. Prove Proposition L.20: Let $A \subset \mathbb{R}^p$. The following are equivalent:

(i) $A \in \mathfrak{M}(\mu)$.

(ii) $\forall \epsilon > 0$, $\exists$ an open $G \supset A$ ($G \in \mathfrak{M}(\mu)$, by L.15) such that $\mu^*(G \setminus A) < \epsilon$.

(iii) $\exists$ a sequence of open sets (G_n) with $A \subset \bigcap_n G_n$ and $\mu^*((\bigcap_n G_n) \setminus A) = 0$.

(iv) $\forall \epsilon > 0$, $\exists$ a closed $F \subset A$ ($F \in \mathfrak{M}(\mu)$ since $\mathfrak{M}(\mu)$ is a ring) such that $\mu^*(A \setminus F) < \epsilon$.

(v) $\exists$ a sequence of closed sets (F_n) with $\bigcup_n F_n \subset A$ and $\mu^*(A \setminus (\bigcup_n F_n)) = 0$.

Hint: First show that any set whose outer measure is zero must be μ-measurable.