MATH 362: Problem Set #21

Do parts (a) and (d) of the following problem:

$\star$33. For each fixed pair $\alpha > 0$, $\delta > 0$, and each $E \subset \mathbb{R}$, set

$$\mu_\delta^*(E) \;:=\; \inf \left\{ \sum_j m(I_j)^\alpha \,|\, \text{each } I_j \text{ is an open interval of } \mathbb{R} \text{ with } m(I_j) \leq \delta, \text{ and } E \subset \bigcup_j I_j \right\}.$$

(Note: We might also refer to the set function μ_δ^* as $\mu_{\delta,\alpha}^*$ when we want to be explicit about its dependence upon the parameter α.)

(a) Show that μ_δ^*, when applied to subsets of $\mathbb{R}$, has properties (i)–(iii) and (v) of Theorem L.6.

(b) Show that, if $\delta > \sigma > 0$, then for each fixed $E \subset \mathbb{R}$, $\mu_\sigma^*(E) \geq \mu_\delta^*(E)$. Then show $\lim_{\delta \to 0^+} \mu_\delta^*(E)$ exists and is equal to $\sup \{\mu_\delta^*(E) \,|\, \delta > 0\}$.

(c) For each $E \subset \mathbb{R}$, set $H_\alpha(E) := \lim_{\delta \to 0^+} \mu_\delta^*(E)$. Show that, like the μ_δ^*, H_α has properties (i)–(iii) and (v) of Theorem L.6 when applied to the subsets of $\mathbb{R}$.

(d) Show that if $H_\alpha(E) < \infty$ and $\beta > \alpha$, then $H_\beta(E) = 0$.

(e) For each $E \subset \mathbb{R}$, set $D(E) := \inf \{\alpha > 0 \,|\, H_\alpha(E) < \infty\}$. (If $\{\alpha > 0 \,|\, H_\alpha(E) < \infty\} = \varnothing$, take $D(E) = \infty$.) Show that if $\beta < D(E)$ then $H_\beta(E) = \infty$, and if $\beta > D(E)$ then $H_\beta(E) = 0$. (Note: $D(E)$ is called the **Hausdorff dimension** of E.)

(f) Show that, for C the usual Cantor middle-thirds set, $D(C) \leq \log(2)/\log(3)$. Show also that $D(\mathbb{R}) = 1$.

(g) Show that the inequality in part (f) is actually equality: $D(C) = \log(2)/\log(3)$.