

MATH 362: Problem Set #17

7.4 You need not answer the question of where the series converges absolutely (we did that one in class). You also need not repeat all steps of the argument we were making concerning application of the Weierstrass M -test (in regards to the question of on what intervals the series is uniformly convergent). However, we did not go so far as to give a complete answer to that question. Pick up where we left off, and determine the answer. Then answer the remaining questions.

8.1 Define

$$f(x) := \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \dots$

8.2 Let a_{ij} be the number in the i^{th} row, j^{th} column of the array

$$\begin{array}{cccccc} -1 & 0 & 0 & 0 & \cdots & \\ \frac{1}{2} & -1 & 0 & 0 & \cdots & \\ \frac{1}{4} & \frac{1}{2} & -1 & 0 & \cdots & \\ \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & -1 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \end{array}$$

so that

$$a_{ij} = \begin{cases} 0, & \text{if } i < j, \\ -1, & \text{if } i = j, \\ 2^{j-1}, & \text{if } i > j. \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \text{but} \quad \sum_j \sum_i a_{ij} = 0.$$

★26. Let X be a set (not necessarily a metric space), and suppose $f_n: X \rightarrow \mathbb{C}$ for each n . Assume also that $f: X \rightarrow \mathbb{C}$. Show that $f_n \rightarrow f$ uniformly on X if and only if $f_n(x_n) - f(x_n) \rightarrow 0$ for all sequences (x_n) in X .

Hint: Prove the “if” part of this claim via contradiction.