

MATH 362: Problem Set #16

Read Sections 7.1–7.10, and 7.19 in Rudin. (I should have assigned this reading last time.)

★24. Suppose that $f: [a, b] \rightarrow \mathbb{C}$ is continuous.

(a) Show that f is uniformly continuous on $[a, b]$. (I realize this result is quite close to Theorem 4.19. I am assigning it so that you go through that proof carefully. Instead of saying “This is true by Theorem 4.19”, I want you to repeat the essential details of the proof as applied in this setting.)

(b) Let \tilde{f} be an extension of f to all of \mathbb{R} defined as follows:

$$\tilde{f}(x) := \begin{cases} f(a), & \text{if } x < a, \\ f(x), & \text{if } a \leq x \leq b, \\ f(b), & \text{if } x > b. \end{cases}$$

Show that \tilde{f} is uniformly continuous on \mathbb{R} .

7.4 Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?

7.8 If

$$I(x) := \begin{cases} 0, & \text{for } x \leq 0, \\ 1, & \text{for } x > 0, \end{cases}$$

if (x_n) is a sequence of distinct points of (a, b) , and if $\sum |c_n|$ converges, prove that the series

$$f(x) := \sum_{n=1}^{\infty} c_n I(x - x_n), \quad a \leq x \leq b,$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

★25. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$f(x) := \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Use induction to show that there exist polynomials $P_0, P_1, \dots, P_n, \dots$, such that $f^{(n)}(x) = P_n(1/x)e^{-1/x^2}$, for all $x \neq 0, n = 0, 1, 2, \dots$