MATH 362: Problem Set #12

Read Sections 3.47–3.51 in Rudin. Then do the following problems.

3.11 Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(d) What can be said about $\displaystyle\sum \frac{a_n}{1+na_n}$ and $\displaystyle\sum \frac{a_n}{1+n^2 a_n}$?

Perhaps a hint is in order concerning the series $\sum a_n/(1+na_n)$. If we set $a_n = 1$ for those $n = 2^m$, $m = 1, 2, \ldots$, and $a_n = 0$ for all other n, then $\sum a_n = \infty$, but (as should be easy to show) $\sum a_n/(1+na_n)$ converges. This does not fit the prescription, as not every $a_n > 0$. But perhaps with some sort of modification to this series, you can produce an example where $\sum a_n$ diverges but $\sum a_n/(1+na_n)$ converges. Such a thing will not, of course, prove that $\sum a_n/(1+na_n)$ *always* converges despite the divergence of $\sum a_n$. It is just a step you might take in your investigations.

⋆20. Prove that there exists a rearrangement of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots ,$$

whose sequence of partial sums converges to $+\infty$.

⋆21. Prove Proposition S.23: Suppose $a_n \geq 0$, $\forall n \in \mathbb{N}$. Then

$$\sum a_n = \sup\left\{\sum_{n\in F} a_n \,\middle|\, F \text{ is a finite subset of } \mathbb{N}\right\} .$$