Exam 2, Take-Home Part Solutions

1. (a) Let $a \in \mathbb{R}$. The set

$$\{x \in X \,|\, g(x) > a\} \;=\; (\{x \,|\, f(x) > a\} \cap \{x \,|\, f(x) = g(x)\}) \cup (\{x \,|\, g(x) > a\} \cap \{x \,|\, g(x) \neq f(x)\}) \;.$$

Since f is measurable, $\{x \,|\, f(x) > a\} \in \mathfrak{M}$. The space X is measurable and splits into the disjoint union

$$X \;=\; \{x \,|\, f(x) = g(x)\} \cup \{x \,|\, f(x) \neq g(x)\} \;,$$

where the set $\{x \,|\, f(x) \neq g(x)\}$ has (outer) measure zero, so it is measurable. (See Exercise $\star$39.) Hence,

$$\{x \,|\, f(x) = g(x)\} \;=\; X \setminus \{x \,|\, f(x) \neq g(x)\}$$

and, as a result,

$$\{x \,|\, f(x) > a\} \cap \{x \,|\, f(x) = g(x)\} \;,$$

are measurable. Moreover,

$$\{x \,|\, g(x) > a\} \cap \{x \,|\, g(x) \neq f(x)\} \;\subset\; \{x \,|\, g(x) \neq f(x)\} \;,$$

Since the set on the right has outer measure zero, so does the one on the left, and hence it is measurable. Thus, $\{x \,|\, g(x) > a\} \in \mathfrak{M}$. Since $a \in \mathbb{R}$ is arbitrary, we have that g is measurable.

(b) Let $t \in X$ be such that $f(t) < g(t)$. Then $\exists q \in \mathbb{Q}$ such that $f(t) < q < g(t)$, which shows that $t \in \{x \,|\, f(x) < q\} \cap \{x \,|\, g(x) > q\}$. Now let $\{q_j\}_{j=1}^{\infty}$ be an enumeration of $\mathbb{Q}$, and set $E_j = \{x \,|\, f(x) < q_j\} \cap \{x \,|\, g(x) > q_j\}$ for $j = 1, 2, \ldots$. Since f, g are measurable, each $E_j \in \mathfrak{M}$. Moreover,

$$\{x \,|\, f(x) < g(x)\} \;=\; \bigcup_{j=1}^{\infty} E_j \;,$$

which is measurable since $\mathfrak{M}$ is closed under countable unions.

By symmetry of argument, the set $\{x \,|\, g(x) < f(x)\}$ is also measurable. But then

$$\{x \,|\, f(x) = g(x)\} \;=\; X \setminus (\{x \,|\, f(x) < g(x)\} \cup \{x \,|\, f(x) > g(x)\})$$

is measurable as well, since $\mathfrak{M}$ is a ring.

2. Let A, $B \in \mathfrak{N}$, which means $f^{-1}(A)$, $f^{-1}(B) \in \mathfrak{M}$. Then

$$f^{-1}(A \cup B) \;=\; f^{-1}(A) \cup f^{-1}(B) \;\in\; \mathfrak{M}$$

and

$$f^{-1}(A \setminus B) \;=\; f^{-1}(A) \setminus f^{-1}(B) \;\in\; \mathfrak{M} \,,$$

showing that $\mathfrak{N}$ is a ring.

Now suppose (A_j) is a sequence of sets in $\mathfrak{N}$. It is the case that

$$f^{-1}\left(\bigcup_j A_j\right) \;=\; \bigcup_j f^{-1}(A_j) \;\in\; \mathfrak{M} \,,$$

since $\mathfrak{M}$ is closed under countable unions. Thus, $\mathfrak{N}$ is a σ-ring.

To get that $\mathfrak{N}$ is a σ-algebra, notice that all sets in $\mathfrak{N}$ are subsets of $[-\infty, \infty]$, and $f^{-1}([-\infty, \infty]) = X \in \mathfrak{M}$, which shows $[-\infty, \infty] \in \mathfrak{N}$.

3. For each open $\Omega \subset \mathbb{R}$, Ω may be written as the countable union of open intervals $I_j = (a_j, b_j)$ ($\mathbb{R}$ has a countable base of open intervals whose endpoints are rational numbers). Moreover,

$$f^{-1}(\Omega) \;=\; f^{-1}\left(\bigcup_j I_j\right) \;=\; \bigcup_j f^{-1}(I_j) \;=\; \bigcup_j (\{x \,|\, f(x) < b_j\} \cap \{x \,|\, f(x) > a_j\}) \,.$$

This shows $f^{-1}(\Omega) = \{x \,|\, f(x) \in \Omega\}$ is measurable for each open $\Omega \subset \mathbb{R}$.

Now let $c \in \mathbb{R}$. Since (c, ∞) is an open interval and g is continuous, we have $\Omega_c := g^{-1}((c, \infty))$ is open in $\mathbb{R}$. Thus,

$$\{x \,|\, (g \circ f)(x) > c\} \;=\; f^{-1}(\Omega_c) \in \mathfrak{M} \,.$$

Since this set is measurable for each $c \in \mathbb{R}$, we have $g \circ f$ is measurable.

4. (a) Suppose $x = 0.a_1a_2a_3a_4\ldots = 0.\gamma_1\gamma_2\gamma_3\gamma_4\ldots$, where each a_j, $\gamma_j \in \{0, 1, 2\}$ and some $a_j \neq \gamma_j$. Then one of these ternary expansions—let us suppose it is the one with γ's—ends in a string of 2's. That is, there is a natural number M for which $\gamma_j = 2$ for each $j > M$, but $\gamma_M \neq 2$. With what we know about expansions, we have that $a_j = \gamma_j$ for $j = 1, \ldots, M-1$, $a_M = 1 + \gamma_M$, and $a_j = 0$ for $j > M$.

   Now let the number $N_a \in \mathbb{N}$ be such that $a_{N_a} = 1$ and $a_j \neq 1$ for $j < N_a$, if such a number N_a exists; otherwise, let $N_a = \infty$. Choose the b_j, $j < N_a$ as described above. Working from the exansion $0.\gamma_1\gamma_2\gamma_3\ldots$ in place of $0.a_1a_2a_3\ldots$, choose N_γ analogously to N_a and numbers β_j, $j < N_\gamma$ analogously to the b_j. Notice that, since $a_j = 0$ for $j > M$, we must have $N_a \leq M$. We consider two cases:

   - Case: $N_a < M$
     Since the $a_j = \gamma_j$ for $j < M$, it is clear that the $N_a = N_\gamma$, and the $b_j = \beta_j$, $j = 1, \ldots, N_a$. Thus, $\sum_{n=1}^{N_a} b_n 2^{-n} = \sum_{j=1}^{N_\gamma} \beta_j 2^{-j}$.
   - Case: $N_a = M$
     Since $\gamma_M \neq a_M - 1 = 0$ and $\gamma_j = 2$ for $j > M$, it follows that $N_\gamma = \infty$, $\beta_M = 0$ and $\beta_j = 1$ for $j > M$. Obviously it is still true that $b_j = \beta_j$ for $j < M$, and $b_M = 1$. Thus,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\beta_n}{2^n} &= \sum_{n=1}^{M-1} \frac{\beta_n}{2^n} + \sum_{n=M+1}^{\infty} \frac{\beta_n}{2^n} = \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \sum_{n=M+1}^{\infty} \left(\frac{1}{2}\right)^n \\
&= \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \frac{1}{2^{M+1}} \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \frac{1}{2^{M+1}} \cdot \frac{1}{1 - (1/2)} \\
&= \sum_{n=1}^{M-1} \frac{b_n}{2^n} + \frac{1}{2^M} = \sum_{n=1}^{M} \frac{b_n}{2^n} \,.
\end{aligned}$$

   (b) As we know, the set $[0, 1] \setminus C$ is made up of countably-many disjoint open intervals I_k^n, $n = 0, 1, 2, \ldots$, $k = 1, \ldots, 2^n$, where

   $I_1^0 = \left(\frac{1}{3}, \frac{2}{3}\right)$,
   $I_1^1 = \left(\frac{1}{3^2}, \frac{2}{3^2}\right)$, $I_2^1 = \left(\frac{7}{3^2}, \frac{8}{3^2}\right)$,
   $I_1^2 = \left(\frac{1}{3^3}, \frac{2}{3^3}\right)$, $I_2^2 = \left(\frac{7}{3^3}, \frac{8}{3^3}\right)$, $I_3^2 = \left(\frac{19}{3^3}, \frac{20}{3^3}\right)$, $I_4^2 = \left(\frac{25}{3^3}, \frac{26}{3^3}\right)$,
   etc.

Fix n and k, and let $x, y \in \overline{I_k^n}$. This means that x, y have ternary expansions $0.a_1a_2a_3\ldots$, $0.c_1c_2c_3\ldots$ respectively such that $a_j = c_j \in \{0, 2\}$ for $j = 1, \ldots, n$, and $a_k = c_k = 1$. Thus,

$$f(x) \;=\; \frac{1}{2^k} + \sum_{m=1}^{k-1} \frac{a_m}{2^{m+1}} \;=\; \frac{1}{2^k} + \sum_{m=1}^{k-1} \frac{c_m}{2^{m+1}} \;=\; f(y)\;,$$

where a sum $\sum_{m=1}^{k-1}$ should be taken as zero if $k = 1$.

An elementary fact about expansions is the following: If $x, y \in [0, 1]$ with $x \neq y$, and $0.x_1x_2x_3\ldots$, $0.y_1y_2y_3\ldots$ are base-b ($b \in \mathbb{N} \setminus \{1\}$) expansions of x and y (each x_j, $y_j \in \{0, 1, \ldots, b-1\}$), then taking M_b be the smallest positive integer j such that $x_j \neq y_j$ (a well-defined number), we have $x < y$ if and only if $x_{M_b} < y_{M_b}$. Thus, let us suppose $x < y$, that they have ternary expansions $0.a_1a_2a_3\ldots$, $0.c_1c_2c_3\ldots$, respectively, and that $M = M_3$ is a positive integer such that $a_j = c_j$ for $j < M$ but $a_M < c_M$. Let N be the smallest j for which $a_j = 1$. ($N = \infty$ if this criterion cannot be met.) If $N > M$, then since $c_M > a_M$, it follows that $a_M = 0$. Now, if $c_M = 1$, then

$$f(y) \;=\; \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots + \frac{a_{M-1}}{2^M} + \frac{1}{2^M}\;,$$

while if $c_M = 2$,

$$f(y) \;=\; \frac{a_1}{2^2} + \frac{a_2}{2^3} + \cdots + \frac{a_{M-1}}{2^M} + \frac{2}{2^{M+1}} + \text{other nonnegative terms}\;.$$

Either way

$$f(x) \;\le\; \frac{a_1}{2^2} + \cdots + \frac{a_1}{2^2} + \sum_{j=M+1}^{\infty} \frac{2}{2^{j+1}} \;=\; \frac{a_1}{2^2} + \cdots + \frac{a_1}{2^2} + \frac{1}{2^M} \;\le\; f(y)\;.$$

If $N \le M$, it is even easier to see that $f(x) \le f(y)$.

Next note that every real number in $[0, 1]$ has a binary expansion $0.b_1b_2b_3\ldots$, each $b_j \in \{0, 1\}$. To find an x such that

$$f(x) \;=\; \sum_{j=1}^{\infty} \frac{b_j}{2^j}\;,$$

take x to be the number with ternary expansion $0.a_1a_2a_3\ldots$ where each $a_j = 2b_j$. By inspection, $f(0) = 0$ and $f(1) = 1$.

Thus far, we have shown that f maps $[0, 1]$ *onto* $[0, 1]$, and is monotone increasing. Being monotone, the only kind of discontinuity f can have is a jump discontinuity. But if it had any such discontinuities, there would be numbers in $[0, 1]$ absent from the range of f, and this is not the case. Thus, f is continuous. To finish, note that we showed f is constant not just on each open interval I_k^n in $[0, 1] \setminus C$, but on the closure $\overline{I_k^n}$. The endpoints of each I_k^n are points of C, and hence there are no members of the range of f not found in the image of C under f.

(c) By part (b), g is continuous and, for $y > x$, $g(y) = f(y)+y \ge f(x)+y > f(x)+x = g(x)$, so g is injective (one-to-one). Since $g(0) = 0$ and $g(1) = 2$, the intermediate value theorem tells us that g maps $[0, 1]$ onto $[0, 2]$, making it a one-to-one correspondence between the compact sets $[0, 1]$ and $[0, 2]$. By Exercise $\star$23, g is an homeomorphism.

(d) Let I_k^n be the open intervals contained in $[0,1]$ defined in the proof of part (b). Since $f = c$, c a constant on each I_k^n (the constant changes for different I_k^n), the set $g(I_k^n) = \{g(x) \,|\, x \in I_k^n\} = c + I_k^n$. Thus, by Lemma L.22, $m(g(I_k^n)) = m(I_k^n) = 3^{-(n+1)}$. We now have

$$[0,2] \;=\; g(C) \cup g\left(\bigcup_{n=0}^{\infty}\bigcup_{k=1}^{2^n} I_k^n\right) \;=\; g(C) \cup \left(\bigcup_n \bigcup_k g(I_k^n)\right) .$$

Thus,

$$g(C) \;=\; [0,2] \setminus \left(\bigcup_{n=0}^{\infty}\bigcup_{k=1}^{2^n} g(I_k^n)\right) \;\in\; \mathfrak{M} ,$$

and, since the $g(I_k^n) \in [0,2]$ are pairwise disjoint,

$$\begin{aligned} m(g(C)) \;&=\; m([0,2]) - m\left(\bigcup_{n,k} g(I_k^n)\right) \;=\; 2 - \sum_{n=0}^{\infty}\sum_{k=1}^{2^n} m(g(I_k^n)) \\ &=\; 2 - \sum_n \sum_k m(I_k^n) \;=\; 2 - \sum_n \sum_k 3^{-(n+1)} \\ &=\; 2 - \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n \;=\; 2 - \frac{1}{3}\cdot\frac{1}{1-2/3} \;=\; 1 . \end{aligned}$$

Since $g(C)$ has positive measure, it contains a subset B which is nonmeasurable. Now take $A = h(B) \subset C$. Since the Cantor set C has measure zero, $m(A) = 0$, and hence $A \in \mathfrak{M}$. However, $B = h^{-1}(A)$ is nonmeasurable.

(e) If we take $p = \chi_A$ (A as defined in the part (d)), and $q = h = g^{-1}$, then $p \circ q$ is not measurable. To see this, suppose $\Omega \subset \mathbb{R}$ is an open set with $1 \in \Omega$ but $0 \notin \Omega$. If $\chi_A \circ h$ were measurable, then it would be the case that $(\chi_A \circ h)^{-1}(\Omega)$ would be measurable (this because Ω is a countable union of sets of the form $[-\infty, b) \cap (a, \infty]$, and the preimage of such sets under a measurable function is measurable). But,

$$(\chi_A \circ h)^{-1}(\Omega) \;=\; \{x \,|\, (\chi_A \circ h)(x) \in \Omega\} \;=\; g(A) \;\notin\; \mathfrak{M} .$$

(f) The set we have that fits these criteria is A (from part (d)). We have already established that A is measurable.

To see that it is not a Borel set, we first make a few observations. We have that $h = g^{-1}$ is continuous, and hence a measurable function. We argued in a previous problem (Problem 2 on the in-class portion of this exam) that $h^{-1}(\Omega) \in \mathfrak{M}$ for each open Ω. Thus, the σ-algebra $\mathfrak{N}$ (associated with h) from Problem 2 (of this take-home portion of the exam) contains all open sets. The definition of the Borel algebra $\mathscr{B}$ is that it is the smallest σ-algebra containing every open set, and hence $\mathscr{B} \subset \mathfrak{N}$.

Now suppose (to get a contradiction) $A \in \mathscr{B}$. Thus, $A \in \mathfrak{N}$, meaning that it is one of the sets for which $h^{-1}(A) = g(A) \in \mathfrak{M}$. $\rightarrow\!\leftarrow$

5. (a) Consider the sets $A_n = E_1 \setminus E_n$, $n \in \mathbb{N}$. Since $E_{n+1} \subset E_n$ for each n, we have $A_n \subset A_{n+1}$. Set $A = \bigcup_n A_n$. We have

$$m(A) \;=\; \lim_n m(A_n) \;=\; \lim_n m(E_1 \setminus E_n) \;=\; \lim_n [m(E_1) - m(E_n)] \;=\; m(E_1) - \lim_n m(E_n) ,$$

where the first equality holds by Theorem 11.3, and our assertion $m(E_1 \setminus E_n) = m(E_1) - m(E_n)$ uses the assumption that $m(E_1) < \infty$. But,

$$A \;=\; \bigcup_n A_n \;=\; \bigcup_n (E_1 \setminus E_n) \;=\; E_1 \setminus \left(\bigcap_n E_n \right) \,,$$

so

$$m(A) \;=\; m(E_1) - m\left(\bigcap_n E_n \right) \,.$$

Thus, we have

$$m(E_1) - m\left(\bigcap_n E_n \right) \;=\; m(E_1) - \lim_n m(E_n) \,,$$

which yields the result.

(b) For each n let $E_n = [n, \infty) \subset \mathbb{R}$. Each $E_{n+1} \subset E_n$ and $m(E_n) = \infty$, $\forall n$, so $\lim_n m(E_n) = \infty$. But $\bigcap_n E_n = \varnothing$.

(c) Take ϵ, $\delta > 0$ to be given. For each $n \in \mathbb{N}$ let

$$G_n \;:=\; \{x \in E \,:\, |f_n(x) - f(x)| \geq \epsilon\} \,, \qquad \text{and set} \qquad E_k \;:=\; \bigcup_{n=k}^{\infty} G_n \,,$$

for $k = 1, 2, \ldots$. The measurability of each f_n implies that the pointwise limit function f is measurable as well. Since each f_n is real-valued, $f_n - f$ is a well-defined function (no instances of $\infty - \infty$) and is measurable, so $|f_n - f|$ is measurable, too. Thus, the sets G_n and E_k are measurable. Note that $E_{k+1} \subset E_k$ for each $k \in \mathbb{N}$, and that $\bigcap_k E_k = \varnothing$ by virtue of the fact that $f_n \to f$ pointwise on E. Finally, $E_1 \subset E$ and so $m(E_1) \leq m(E) < \infty$. Applying part (a), we have

$$\lim_{k\to\infty} m(E_k) \;=\; m\left(\bigcap_{k=1}^{\infty} E_k \right) \;=\; m(\varnothing) \;=\; 0 \,.$$

Thus, $\exists N \in \mathbb{N}$ such that $k \geq N$ implies $m(E_k) < \delta$. By definition, $x \in E_N$ means $x \in G_n$ for some $n \geq N$, and hence there is some $n \geq N$ for which $|f_n(x) - f(x)| \geq \epsilon$. By taking $A = E_N$, we get that for $x \notin A$, $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$.

(d) The first issue is "What f should we take to be the limit function?" Let $S = \{x \in E \,|\, (f_n(x))$ is a converg
We may define

$$f(x) \;:=\; \begin{cases} \lim_n f_n(x) \,, & \text{if } x \in S \,, \\ 0 \,, & \text{if } x \in E \setminus S \,. \end{cases}$$

One can show that f is measurable (reliant on the fact that $m(E \setminus S) = 0$). By problem 1(a), no matter how f is defined on $E \setminus S$, it will still be measurable.

We may define the sets G_n, E_k as in the previous proof, and argue they are measurable just as before. Once again, $E_{k+1} \subset E_k$ for $k = 1, 2, \ldots$, but this time $\bigcap_k E_k = E \setminus S$, so we get

$$\lim_{k\to\infty} m(E_k) \;=\; m\left(\bigcap_{k=1}^{\infty} E_k \right) \;=\; m(E \setminus S) \;=\; 0 \,.$$

The rest of the proof is identical.

(e) Let $\eta > 0$ be given. For $n = 1, 2, \ldots$, let $A_n \subset E$ be chosen as in part (d) so that $m(A_n) < 2^{-n}\eta$ and $\exists N_n \in \mathbb{N}$ such that $k \geq N_n$ and $x \notin A_n$ imply $|f_k(x) - f(x)| < \frac{1}{n}$. Let $A = \bigcup_n A_n$. Then since each $A_n \in \mathfrak{M}$, A is measurable and

$$m(A) \leq \sum_{n=1}^{\infty} m(A_n) \leq \eta \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \eta .$$

To see that $f_n \to f$ uniformly on $E \setminus A = E \setminus \bigcup_n A_n = \bigcap_n (E \setminus A_n)$, let $\epsilon > 0$. For some $N \in \mathbb{N}$ it is the case $0 < \frac{1}{N} < \epsilon$. The set $E \setminus A \subset E \setminus A_N$, so $k \geq N$ implies $|f_k(x) - f(x)| < \epsilon$ for every $x \in E \setminus A$.