Exam 2: Solutions, In-Class Part

1. Continuity is a local property. If we choose an $x \in (-R, R)$, there is a compact subset $[a, b]$ of $(-R, R)$ that contains x. By Theorem 8.1, we have that the power series converges uniformly on compact subsets of $(-R, R)$—in particular, on $[a, b]$. The sequence (s_n) of partial sums

$$s_n(x) \ := \ \sum_{j=0}^{n} a_j x^j \ ,$$

is made up of polynomials, so each s_n is continuous on $[a, b]$. By uniform convergence of the s_n to f on $[a, b]$, and by Corollary 7.12 ("the uniform limit of continuous functions is continuous"), we get that f is continuous on $[a, b]$. In particular, it is continuous at x.

2. To prove the "if" part, we assume that $f^{-1}(\Omega) \in \mathfrak{M}$ for each open $\Omega \subset [-\infty, \infty]$. In particular, this means that $f^{-1}((a, \infty]) = \{x \in X \,|\, f(x) > a\} \in \mathfrak{M}$ for each $a \in \mathbb{R}$, giving that f (by definition) is measurable.

To prove the "only if" part, we first note that, for $a < b$,

$$f^{-1}((a, b)) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty]) \, \{x \,|\, f(x) < b\} \cap \{x \,|\, f(x) > a\} \ ,$$

showing that $f^{-1}(I) \in \mathfrak{M}$ for each open interval $I \subset [-\infty, \infty]$, by Theorem 11.15 and the fact that rings are closed under intersections. To finish, we use that fact that $\mathbb{R}$ has a countable base of open intervals (say, ones whose endpoints are rational numbers or $\pm\infty$). Thus, given an open $\Omega \subset \mathbb{R}$, there exists a sequence of open intervals (I_n) such that $\Omega = \bigcup_n I_n$. Then

$$f^{-1}(\Omega) \ = \ f^{-1}\left(\bigcup_n I_n\right) \ = \ \bigcup_n f^{-1}(I_n) \ ,$$

again in $\mathfrak{M}$ since σ-algebras are closed under countable unions.

3. (a) Let $A_1 = A$ (the set in $\mathfrak{M}$ for which $\varphi(A) < \infty$, and let $A_n = \varnothing$ for $n = 2, 3, \ldots$. Then the sets $(A_n)_{n=1}^{\infty}$ are pairwise disjoint with union A. By countable additivity,

$$\varphi(A) \ = \ \varphi(A) + \sum_{n=2}^{\infty} \varphi(\varnothing) \ , \qquad \text{or} \qquad \sum_{n=2}^{\infty} \varphi(\varnothing) \ = \ 0 \ .$$

(Note the need for $\varphi(A) < \infty$.) Thus, $\varphi(\varnothing) = 0$.

(b) Let A, B be disjoint members of $\mathfrak{M}$, take $A_1 = A$, $A_2 = B$, and $A_n = \varnothing$ for $n = 3, 4, \ldots$. By countable additivity, we have

$$\varphi(A \cup B) \ = \ \varphi\left(\bigcup_n A_n\right) \ = \ \phi(A) + \phi(B) + \sum_{n=3}^{\infty} \varphi(\varnothing) \ = \ \phi(A) + \phi(B) \ ,$$

where this latter equality holds by part (a).

(c) Let A, $B \in \mathfrak{M}$, with $A \subset B$. By part (b),

$$\varphi(B) \;=\; \varphi(A \cup (B \setminus A)) \;=\; \varphi(A) + \varphi(B \setminus A) \;\geq\; \varphi(A)\;,$$

with the latter inequality holding since φ is a nonnegative set function (so $\varphi(B \setminus A) \geq 0$).

(d) Set $B_1 = E_1$, $B_2 = E_1 \setminus E_2$, and, $B_n = E_n \setminus (E_1 \cup \cdots \cup E_{n-1})$ for $n = 3, 4, \ldots$. Then each $B_j \in \mathfrak{M}$, $B_i \cap B_j = \varnothing$ for $i \neq j$, and $\bigcup_j B_j = \bigcup_n E_n$. Thus,

$$\varphi\left(\bigcup_n E_n\right) \;=\; \varphi\left(\bigcup_n B_n\right) \;=\; \sum_n \varphi(B_n) \;\leq\; \sum_n \varphi(E_n)\;,$$

where the second equality holds by countable additivity of φ, and the inequality holds by part (c).

4. Let (E_j) be a sequence of pairwise disjoint sets in $\mathfrak{M}$. For each $n \in \mathbb{N}$, the additivity of μ implies that

$$\sum_{j=1}^{n} \mu(E_j) \;=\; \mu\left(\bigcup_{j=1}^{n} E_j\right) \;\leq\; \mu\left(\bigcup_{j=1}^{\infty} E_j\right)\;,$$

where the inequality holds by the monotonicity of μ. Since the right-hand expression is an upper bound for the sequence of partial sums of the infinite series $\sum_j \mu(E_j)$, and since μ is nonnegative, it follows that

$$\sum_{j=1}^{\infty} \mu(E_j) \;\leq\; \mu\left(\bigcup_{j=1}^{\infty} E_j\right)\;.$$

But the sub σ-additivity of μ is precisely the opposite inequality, yielding the result.

5. (a) First, note that for t, $s \in \mathbb{R}$,

$$|G(t) - G(s)| \;=\; \sqrt{[x(t) - x(s)]^2 + [y(t) - y(s)]^2} \;\leq\; |x(t) - x(s)| + |y(t) - y(s)|\;.$$

(See Theorem M.4.) Thus, to get that G is continuous it suffices to show that x and y are both continuous functions (of t) on $\mathbb{R}$. (That is, if $\epsilon > 0$ is given, and if, for t fixed, it is possible to find a $\delta > 0$ for which

$$|t - s| \;<\; \delta \qquad \text{implies} \qquad |x(t) - x(s)| \;<\; \frac{\epsilon}{2} \quad \text{and} \quad |y(t) - y(s)| \;<\; \frac{\epsilon}{2}\;,$$

then this δ suffices to give $|G(t) - G(s)| < \epsilon$.) Now note that $\phi_n(t) := 2^{-n} f(3^{2n-1} t)$ is continuous for each n, being the composition of continuous functions. Moreover, for each $t \in \mathbb{R}$ we have $|\phi_n(t)| \leq 2^{-n}$, since $0 \leq f(u) \leq 1$, $\forall u \in \mathbb{R}$. Since $\sum_n 2^{-n}$ converges, the series $\sum_n \phi_n(t)$ converges uniformly on $\mathbb{R}$. That $x = x(t)$ is continuous on $\mathbb{R}$ now follows from Corollary 7.12. The demonstration that $y = y(t)$ is continuous is almost identical.

(b) To see that G is surjective onto I^2, let $(x_0, y_0) \in I^2$ be fixed. Both x_0, y_0 have a binary expansions $0.c_1c_2c_3\ldots$, $0.d_1d_2d_3\ldots$; that is, there are sequences (c_n), (d_n) with each c_n, $d_n \in \{0, 1\}$ such that

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} c_n \qquad \text{and} \qquad y_0 = \sum_{n=1}^{\infty} 2^{-n} d_n \ .$$

Now let $(a_n)_{n=1}^{\infty}$ be the sequence given by

$$a_n := \begin{cases} c_{(n+1)/2} \ , & \text{if } n \text{ is odd,} \\ d_{n/2} \ , & \text{if } n \text{ is even,} \end{cases}$$

and take

$$t_0 = \sum_{n=1}^{\infty} 3^{-n-1}(2a_n) \ .$$

Note that the sequence $(2a_n)$ consists of numbers in $\{0, 2\}$, and simply represents the digits of the ternary expansion for the number t_0, so this series converges and $t_0 \in [0, 1]$ (actually in $[0, 1/3]$). Now, for $k \in \mathbb{N}$,

$$3^k t_0 = \sum_{n=1}^{k-1} 3^{k-n-1}(2a_n) + \sum_{n=k}^{\infty} 3^{k-n-1}(2a_n) = 2N + \frac{2}{3}a_k + t_k \ ,$$

where N is an integer, and

$$t_k := \sum_{n=k+1}^{\infty} 3^{k-n-1}(2a_n) \leq 2 \sum_{n=k+1}^{\infty} \frac{1}{3^{n+1-k}} = \frac{1}{3} \ .$$

Using the properties assumed for f, we have $f(3^k t_0) = f(\frac{2}{3}a_k + t_k) = a_k$, since $a_k = 0$ or 1 and, hence, $\frac{2}{3}a_k + t_k$ is in $[0, 1/3]$ or $[2/3, 1]$. Thus,

$$x(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1} t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = \sum_{n=1}^{\infty} 2^{-n} c_n = x_0 \ ,$$

and likewise $y(t_0) = y_0$.