MATH 362 Exam 1 Solutions

1. That $\liminf s_n$, $\limsup s_n \in E$ is immediate from Thm. S.9. Thus, we already have $\limsup s_n \leq \sup E$. To prove they must be equal, we consider three cases.
Case $\limsup s_n = +\infty$.
This case is clear.
Case $\limsup s_n = -\infty$.
It follows from Thm. S.6 that $\liminf s_n = \limsup s_n$. So, by Thm. S.7, $s_n \to -\infty$. That there cannot be a subsequence (s_{n_k}) of (s_n) converging to some $s \neq -\infty$ follows since, if there were, then $\forall\, m$ we would have
$$\sup\{s_n \,|\, n \geq m\} \;\geq\; \sup\{s_{n_j} \,|\, j \geq m\} \;\geq\; s \;\Rightarrow\; \limsup s_n \geq s. \qquad \rightarrow\!\leftarrow$$
Case $\limsup s_n = M \in \mathbb{R}$.
Suppose $\exists\, x \in E$ such that $M < x$. Choose y so that $M < y < x$. Thm. S.8 says that $\exists\, N \in \mathbb{N}$ such that $s_n < y$ whenever $n \geq N$. But then the interval $(y, 2x - y)$ contains only finitely many terms of the sequence (s_n), contradicting that s is a limit point of some subsequence (s_{n_j}) of (s_n).

2. (a) Clearly S is bounded, as it is contained in $B(0, 2)$. Next, we note that the interior of S is $B(0, 1)$, a ball in $\mathbb{R}^3$ (hence open). For $|x| > 1$, let $r = (|x| - 1)/2$. Then $\forall\, y \in B(x, r)$, we have
$$|x| \;\leq\; |y| + |x - y| \;\leq\; |y| + \frac{|x| - 1}{2} \quad\Rightarrow\quad |y| \;\geq\; |x| - \frac{1}{2}|x| + \frac{1}{2} \;=\; \frac{1}{2}(|x| + 1) > 1.$$
So, $B(x, r) \cap S = \varnothing$, showing $\{x \in \mathbb{R}^3 \colon |x| > 1\}$ to be an open set. Thus, $\mathbb{R}^3 \setminus S = B(0, 1) \cup \{x \in \mathbb{R}^3 \colon |x| > 1\}$ is open. Hence S is closed. Since it is a closed and bounded subset of Euclidean space, S is compact.
(b) Since Ω is open, Ω^c is closed, having empty intersection with S. By a previous homework problem, we know $\gamma := \operatorname{dist}(\Omega^c, S)/2 > 0$. Thus, $\forall\, x \in S$, $B(x, \gamma) \cap \Omega^c = \varnothing$; that is, $B(x, \gamma) \subset \Omega$. So, take $s = 1 + \gamma$ and $r = \min\{1/2, 1 - \gamma\}$. Then $\{x \in \mathbb{R}^3 \colon r < |x| < s\} \subset \Omega$.

3. Let $n \in \mathbb{N}$. For each $k \geq n$, $a_k b_k \leq (\sup\{a_j \,|\, j \geq n\})(\sup\{b_j \,|\, j \geq n\})$. (Note that the nonnegativity of the a_n, b_n is instrumental here.) Thus,
$$\sup\{a_j b_j \,|\, j \geq n\} \;\leq\; (\sup\{a_j \,|\, j \geq n\})(\sup\{b_j \,|\, j \geq n\}).$$
Taking limits as $n \to \infty$ of both sides, we get
$$\begin{aligned} \limsup_n a_n b_n &\leq \lim_{n\to\infty} \left[(\sup\{a_j \,|\, j \geq n\})(\sup\{b_j \,|\, j \geq n\})\right] \\ &= \left(\lim_{n\to\infty} \sup\{a_j \,|\, j \geq n\}\right)\left(\lim_{n\to\infty} \sup\{b_j \,|\, j \geq n\}\right) \\ &= \left(\limsup_n a_n\right)\left(\limsup_n b_n\right). \end{aligned}$$
To see that inequality may hold here, let
$$a_n := \begin{cases} 0, & \text{if } n \text{ is odd,} \\ 1, & \text{if } n \text{ is even,} \end{cases} \qquad \text{and} \qquad b_n := \begin{cases} 1, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

4. By the last problem,

$$\limsup_n |a_n b_n|^{1/n} \le \left(\limsup_n |a_n|^{1/n}\right)\left(\limsup_n |b_n|^{1/n}\right). \tag{1}$$

If the factors at right in (1) are both positive reals, then this immediately gives us that

$$\limsup_n |a_n b_n|^{1/n} \le \frac{1}{R_a R_b} \qquad \Rightarrow \qquad R := \frac{1}{\limsup_n |a_n b_n|^{1/n}} \ge R_a R_b,$$

where R is the radius of convergence of the power series $\sum a_n b_n z^n$. Other cases may be considered, with results as described in the table:

$\limsup \|a_n\|^{1/n}$	$\limsup \|b_n\|^{1/n}$	product in (1)	$\limsup \|a_n b_n\|^{1/n}$	R
real	0	0	0	∞
0	real	0	0	∞
0	0	0	0	∞
positive real	$+\infty$	$+\infty$	$\in [0, \infty]$	≥ 0
$+\infty$	positive real	$+\infty$	$\in [0, \infty]$	≥ 0
$+\infty$	$+\infty$	$+\infty$	$\in [0, \infty]$	≥ 0

In each of these cases, it is still true that

$$R \ge R_a R_b.$$

The one case in which we may draw no conclusion is if one of $\limsup |a_n|^{1/n}$, $\limsup |b_n|^{1/n}$ is 0 while the other is $+\infty$.

5. (a) See Theorem 1.35 in Rudin's text.

   (b) By the Schwarz inequality,

$$\sum_{j=1}^{N} \sqrt{a_n} \cdot \frac{1}{n} \le \left[\left(\sum_{j=1}^{N} a_n\right)\right]^{1/2} \left[\left(\sum_{n=1}^{N} \frac{1}{n^2}\right)\right]^{1/2}$$

$$\le \left[\left(\sum_{j=1}^{\infty} a_n\right)\right]^{1/2} \left[\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)\right]^{1/2} \le \infty .$$

   This shows that the partial sums of $\sum_n \sqrt{a_n}/n$ are bounded above. Hence the series $\sum_n \sqrt{a_n}/n$ converges by S.1.