MATH 335: Numerical Analysis

Problem Set 15, Solutions

★28 First, we need a vector containing interpolated temperatures. The code below produces such a vector, with the first entry corresponding to a depth of 0 meters, the last 27.25 meters, and one entry for each quarter meter in between.

```
x = [0 2.3 4.9 9.1 13.7 18.3 22.9 27.2];
y = [22.8 22.8 22.8 20.6 13.9 11.7 11.1 11.1];
temps = myspl3(x, y, 0:.25:27.25);
```

In what follows, I will approximate $f'(x)$ and $f''(x)$ using the centered divided difference formulas

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \qquad \text{and} \qquad f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} .$$

The following commands produce approximations of the first derivative of temperature at depths $0.25, 0.5, \ldots, 27$ meters:

```
for j = 2:(length(temps)-1)
  d1temps(j-1) = (temps(j+1)-temps(j-1))/.5;
end
```

Employing the `diff()` command in Octave (a convenient way of getting a vector of values $f(x+h) - f(x)$ in one shot), we get approximations to the 2^{nd} derivative of temperature at these same depths:

```
difftemps = diff(temps);
n = length(difftemps);
d2temps = (difftemps(2:n) - difftemps(1:n-1)) / .25^2;
```

(Actually,

```
d2temps = diff(temps, 2) / .25^2;
```

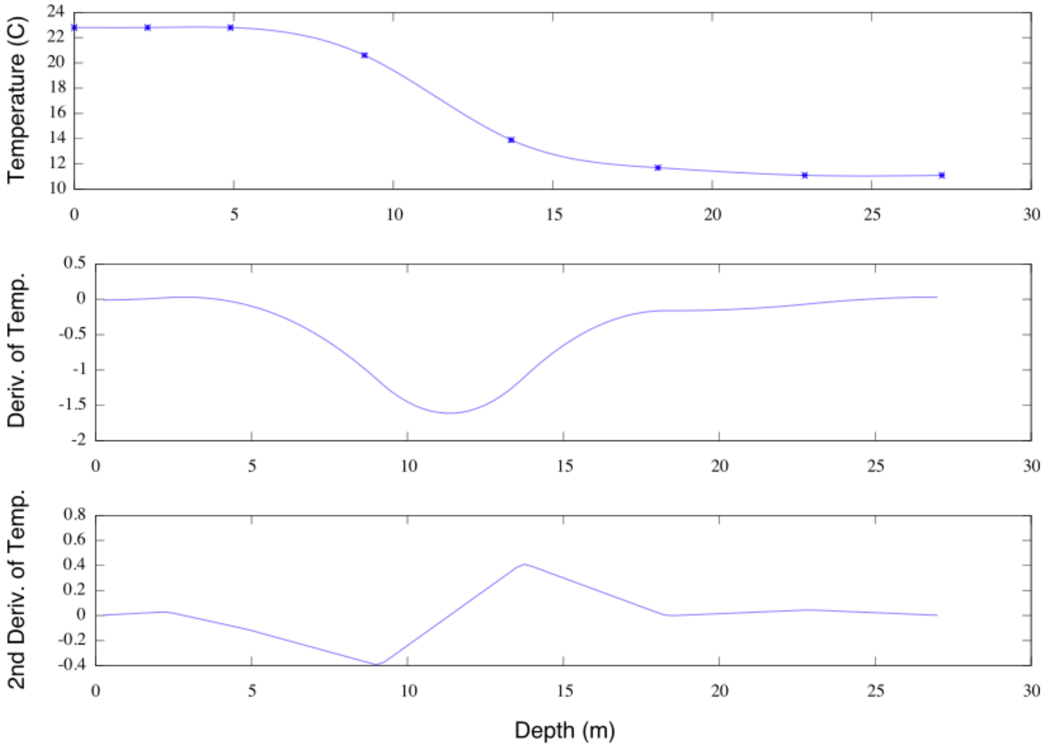
would have done the same thing.) To get the plot below, I executed the commands

```
subplot(3,1,1)
plot(0:.25:27.25, temps, 'b-', x, y, 'b*', 'markersize', 4)
ylabel('Temperature (C)', 'fontsize', 18)

subplot(3,1,2)
plot(.25:.25:27, d1temps, 'b-')
ylabel('Deriv. of Temp.', 'fontsize', 18)

subplot(3,1,3)
plot(.25:.25:27, d2temps, 'b-')
ylabel('2nd Deriv. of Temp.', 'fontsize', 18)
xlabel('Depth (m)', 'fontsize', 18)
```

Visually it appears temperature has a point of inflection around a depth of 11 m; the values of the approximate 2^{nd} derivative change sign between 11.25 and 11.5 m, when the temperature is approximately 17.3°C.



★29 (a) We are solving for zeros of the function $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}(\alpha,\beta) := \begin{bmatrix} \ell\cos\alpha + m\cos\beta - a \\ \ell\sin\alpha + m\sin\beta - b \end{bmatrix}, \qquad \text{with Jacobian} \qquad \mathbf{J_F}(\alpha,\beta) := \begin{bmatrix} -\ell\sin\alpha & -m\sin\beta \\ \ell\cos\alpha & m\cos\beta \end{bmatrix}.$$

```
octave:22> function v = f(angles)
> alp = angles(1); bet = angles(2);
> v(1) = 2*cos(alp) + cos(bet) - 1;
> v(2) = 2*sin(alp) + sin(bet) - 1;
> v = v(:);    % This ensures v is a column vector
> end

octave:23> function J = Jay(angles)
> alp = angles(1); bet = angles(2);
> J = [-2*sin(alp) -sin(bet); 2*cos(alp) cos(bet)];
> end

octave:31> x = [0; pi/2];
octave:32> for j = 1:7, x = x - Jay(x) \ f(x); end, x
x =
   0.29870
   2.71756

octave:33> x = [pi/2; 0];
octave:34> for j = 1:7, x = x - Jay(x) \ f(x); end, x
x =
   1.2721
  -1.1468
```

The configuration $(\alpha, \beta) = (0.2987, 2.71756)$ corresponds to

$$\mathbf{v}_1 \doteq 2\begin{bmatrix}\cos(0.2987)\\ \sin(0.2987)\end{bmatrix} \doteq \begin{bmatrix}1.91144\\ 0.58856\end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 \doteq \begin{bmatrix}\cos(2.71756)\\ \sin(2.71756)\end{bmatrix} \doteq \begin{bmatrix}-0.91144\\ 0.41144\end{bmatrix}.$$

Perhaps not surprisingly due to symmetry, the configuration $(\alpha, \beta) = (1.2721, -1.1468)$ corresponds to

$$\mathbf{v}_1 \doteq 2\begin{bmatrix}\cos(1.2721)\\ \sin(1.2721)\end{bmatrix} \doteq \begin{bmatrix}0.58856\\ 1.91144\end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 \doteq \begin{bmatrix}\cos(-1.1468)\\ \sin(-1.1468)\end{bmatrix} \doteq \begin{bmatrix}0.41144\\ -0.91144\end{bmatrix}.$$

(b) First, taking $\ell = 2$, $m = 1$, we have

$$\begin{aligned}\mathbf{J}_{\mathbf{F}}^{-1}(\alpha, \beta) &= \frac{1}{2(\cos\alpha\sin\beta - \cos\beta\sin\alpha)}\begin{bmatrix}\cos\beta & \sin\beta\\ -2\cos\alpha & -2\sin\alpha\end{bmatrix}\\ &= \frac{1}{2\sin(\beta - \alpha)}\begin{bmatrix}\cos\beta & \sin\beta\\ -2\cos\alpha & -2\sin\alpha\end{bmatrix}.\end{aligned}$$

Thus,

$$\mathbf{G}(\alpha,\beta) \;=\; \begin{bmatrix}\alpha\\ \beta\end{bmatrix} - \frac{1}{2\sin(\beta-\alpha)}\begin{bmatrix}\cos\beta & \sin\beta\\ -2\cos\alpha & -2\sin\alpha\end{bmatrix}\begin{bmatrix}2\cos\alpha+\cos\beta-a\\ 2\sin\alpha+\sin\beta-b\end{bmatrix}.$$

According to Mathematica,

$$\mathbf{J}_{\mathbf{G}}(\alpha,\beta) \;=\; \frac{\csc^2(\alpha-\beta)}{4}\begin{bmatrix}\cos\alpha+\cos(\alpha-2\beta)-2\cos(\alpha-\beta)-2\cos(2(\alpha-\beta))+\sin\alpha-\sin(\alpha-2\beta)-2 & 4-2\cos\alpha+2\cos(\alpha-\beta)-2\sin\alpha\\ 2+4\cos(\alpha-\beta)-2\cos\beta-2\sin\beta & 2\cos(\alpha-\beta)[\cos\alpha+\sin\alpha-2]-2\cos(2(\alpha-\beta))\end{bmatrix}.$$

Thus, $\rho(\mathbf{J}_{\mathbf{G}}(.2987, 2.71756)) \doteq 0.5$ and $\rho(\mathbf{J}_{\mathbf{G}}(1.2721, -1.1468)) \doteq 0.5$ which indicates, by a result given in class, that for all (α,β) sufficiently close to either of these points, Newton's method will converge quadratically.

$\underline{\star 30}$ We have

$$(\cos x + i\sin x)^n \;=\; (e^{ix})^n \;=\; e^{inx} \;=\; \cos(nx) + i\sin(nx)\,.$$

$\underline{\star 31}$ (a) My algorithm looks like:

```
function intApprox = simpson (f, a, b, n)
% function intApprox = simpson (f, a, b, n)
%
%  This routine determines the composite Simpson rule approximation
%  of the integral of f from a to b, using n uniform subintervals.
%
%  INPUTS:
%    f     Handle of the integrand function
%    a, b  Limits of integration
%    n     Number of uniform subintervals to use within interval [a,b]
%
%  OUTPUT: The approximate value of the definite integral

  if (n > 1)
    h = (b - a) / n;
    twoRules = mod(n, 2);
    intApprox = f(a) + f(b - 3*twoRules*h);
    for jj = 0:((n - 3*twoRules)/2 - 1)
      if (jj == 0)
        intApprox += 4*f(a + h);
      else
        intApprox += 2*f(a + 2*jj*h) + 4*f(a + (2*jj+1)*h);;
```

```
      end
    end
    intApprox = intApprox * h/3;
    if (twoRules)
      intApprox += (f(b-3*h) + 3*f(b-2*h) + 3*f(b-h) + f(b)) * 3*h/8;
    end
  else
    disp ('Input n must be a positive integer greater than 1.')
  end
end
```

(c) The integral is improper because of the discontinuity at the lower limit of integration. But, that is not, by itself, the reason for the error. It is also that our algorithm calculates a sum $\sum_{j=0}^{n} w_j f(x_j)$ that specifically chooses $x_0 = a$, precisely where the singularity occurs.

★32 (a) We have

$$\hat{\mathrm{f}}_1 \;=\; \frac{1}{4}\left[(f_0 - f_2) - i(f_1 - f_3)\right]\,, \quad \hat{\mathrm{f}}_2 \;=\; \frac{1}{4}\left[(f_0 + f_2) - (f_1 + f_3)\right]\,,$$

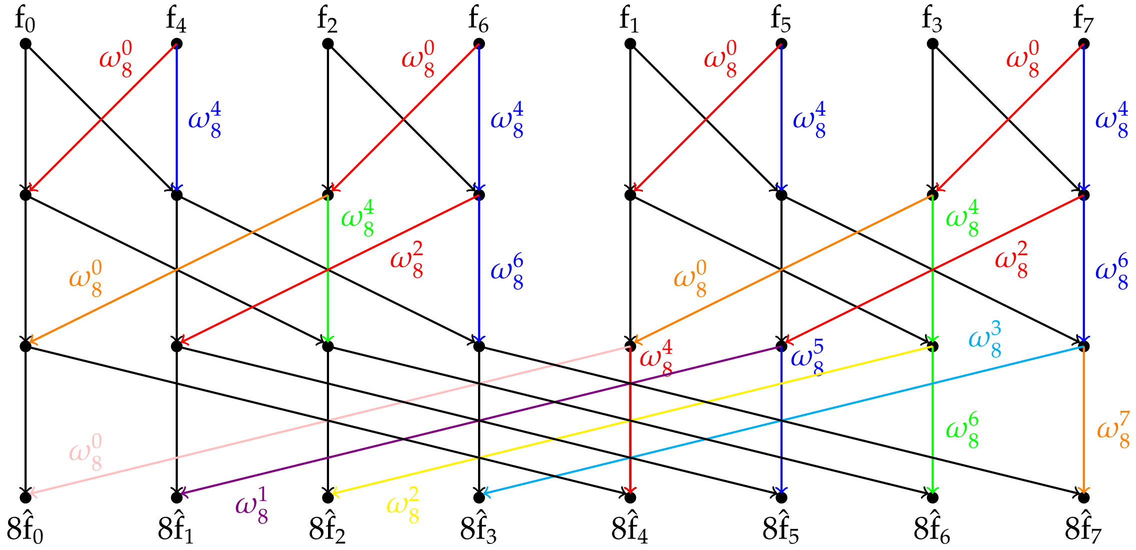
and

$$\hat{\mathrm{f}}_3 \;=\; \frac{1}{4}\left[(f_0 - f_2) + i(f_1 - f_3)\right]\,.$$

(b) Here, we have

$$\hat{\mathrm{f}}_0 \;=\; \frac{1}{2}(f_0 + f_1)\,, \qquad \text{and} \qquad \hat{\mathrm{f}}_1 \;=\; \frac{1}{2}(f_0 - f_1)\,.$$

(c)



(d) From the diagram we have

$$\begin{aligned}
8\hat{\mathrm{f}}_3 &= \mathrm{f}_0 + \omega_8^4\mathrm{f}_4 + \omega_8^6(\mathrm{f}_2 + \omega_8^4\mathrm{f}_6) + \omega_8^3[\mathrm{f}_1 + \omega_8^4\mathrm{f}_5 + \omega_8^6(\mathrm{f}_3 + \omega_8^4\mathrm{f}_7)] \\
&= \mathrm{f}_0 - \mathrm{f}_4 + i(\mathrm{f}_2 - \mathrm{f}_6) - \frac{1}{\sqrt{2}}(1+i)[\mathrm{f}_1 - \mathrm{f}_5 + i(\mathrm{f}_3 - \mathrm{f}_7)] \\
&= 1 - 5 + i(3-7) - \frac{1}{\sqrt{2}}(1+i)[2 - 6 + i(4-8)] \\
&\doteq -4 + 1.6569i\ . \\
\Rightarrow \quad \hat{\mathrm{f}}_3 &\doteq -0.5 + 0.20711i\ .
\end{aligned}$$