MATH 335: Numerical Analysis

Problem Set 14, Solutions

★26 (a) The requirements of a norm $\|\cdot\|$ are

(i) $\|f\| \geq 0$ for all f,

(ii) $\|f\| = 0$ if and only if $f = 0$,

(iii) $\|cf\| = |c|\,\|f\|$ for all f and scalars c, and

(iv) $\|f + g\| \leq \|f\| + \|g\|$.

If we defined

$$|||f|||_n \;:=\; \sqrt{\langle\langle f, f\rangle\rangle_n} \;=\; \sum_{j=0}^{n} f(x_j)\overline{f(x_j)} \;=\; \left(\sum_{j=0}^{n} |f(x_j)|^2\right)^{1/2} ,$$

then, given any function f which satisfied $f(x_j) = 0$ for each $j = 0, \ldots, n-1$, $|||f|||_n = 0$. There are many continuous functions which are not the (constant) zero function yet have value 0 at the n points $x_0, \ldots, x_{n-1}$. Thus, $|||\cdot|||_n$ is not a norm on $C([a, b])$, as it violates property (ii). It might be called a **seminorm**, as it does have each of the other properties.

(b) This one is really a new way to write a fact we already discussed. Or, at least, it will be if we simply take the (new) inner product of both sides of the expression above for f with $E_j(\cdot)$:

$$f = \sum_{k=0}^{n-1} c_k E_k \qquad \Rightarrow \qquad \left\langle\!\left\langle f, E_j \right\rangle\!\right\rangle_n = \left\langle\!\left\langle \sum_{k=0}^{n-1} c_k E_k, E_j \right\rangle\!\right\rangle_n = \sum_{k=0}^{n-1} c_k \left\langle\!\left\langle E_k, E_j \right\rangle\!\right\rangle_n$$

$$= \sum_{k=0}^{n-1} c_k \cdot \frac{1}{n} \sum_{\ell=0}^{n-1} E_k(x_\ell)\overline{E_j(x_\ell)}$$

$$= \sum_{k=0}^{n-1} c_k \left\langle \mathbf{e}^{(k)}, \mathbf{e}^{(j)} \right\rangle_n = \sum_{k=0}^{n-1} c_k \delta_{kj}$$

$$= c_j ,$$

where the $\mathbf{e}^{(j)} = (E_j(x_0), \ldots, E_j(x_{n-1}))$ are defined just like in the handout, and are shown (on that handout) to be *orthonormal*.

★27 I have produced the plot given below using the following commands (which most certainly include ones you have not seen before, but would not necessarily need to draw the same conclusions as I do).

```
load elnino.dat
elnino = elnino';
sf = 1;
n = length(elnino);
t = (1:n);
c = polyfit(t, elnino, 1);
trend = polyval(c, t);
y = elnino - trend;
Y = fft(y);
f = (0:n/2)*sf/n;
pow = abs(Y(1:n/2+1));
figure(1)
subplot(2,1,1)
plot([f;f], [0*pow;pow], 'r-', f, pow, 'b*', 'linewidth', 2, 'markersize', 3)
xlabel('frequency (cycles/month)', 'fontsize', 18)
ylabel('power', 'fontsize', 18)
title('El Nino Periodogram', 'fontsize', 24)
k = 0:20;
zf = k/n;
zpow = pow(k+1);
subplot(2,1,2)
plot([zf;zf], [0*zpow;zpow], 'r-',zf,zpow,'b*','linewidth',2,'markersize',4)
newk = 2:2:18;
newf = newk/n;
period = 1./newf;
for jj = 1:length(period)
  if (jj == 1)
periods = sprintf('%5.1f',period(jj));
  else
periods = [periods; sprintf('%5.1f',period(jj))];
  end
end
set(gca,'xtick',newf)
set(gca,'xticklabel',periods)
xlabel('Period (months/cycle)', 'fontsize', 18)
ylabel('power', 'fontsize', 18)
title('El Nino Periodogram Detail', 'fontsize', 24)
```

The bottom plot gives greater detail by zooming in on the region of interest. To make the bottom plot a little more quirky (but, I think, also useful), I have labeled things with the corresponding period instead of frequency (even though it is still a *frequency* axis). We see the strongest component has period 12 months—this reflects the usual annual changes to the Southern Oscillation Index. But there are other consequential components, the strongest of which seem to have period in the range of 28-42 months. These components may well reflect cyclical changes which we call *el Niño*.