MATH 335: Numerical Analysis
Problem Set 13, Solutions

⋆23 (a) Let's define \( \phi_0(x) = 1, \phi_1(x) = x \). Since we are integrating over an interval that is symmetric around zero, and \( \phi_0, \phi_1 \) are even, odd functions respectively, \( \langle \phi_0, \phi_1 \rangle = 0 \). Next, we have

\[
\begin{align*}
\langle \phi_0, \phi_0 \rangle &= \int_{-1}^{1} 1 \, dx = 2, \\
\langle \phi_1, \phi_1 \rangle &= \int_{-1}^{1} x^2 \, dx = \frac{1}{3} x^3|_{-1}^{1} = \frac{2}{3}, \\
\langle f, \phi_0 \rangle &= \int_{-1}^{1} e^x \, dx = e^x|_{-1}^{1} = e - e^{-1} \approx 2.3504, \\
\langle f, \phi_1 \rangle &= \int_{-1}^{1} xe^x \, dx = (x-1)e^x|_{-1}^{1} = 2e^{-1} \approx 0.73576.
\end{align*}
\]

Thus, we have the matrix system

\[
\begin{bmatrix}
2 & 0 \\
0 & 2/3
\end{bmatrix}
\begin{bmatrix}
b \\
m
\end{bmatrix}
= \begin{bmatrix}
e - e^{-1} \\
2e^{-1}
\end{bmatrix}
\Rightarrow
b = (e - e^{-1})/2 \approx 1.1752;,

m = 3e^{-1} = 1.1036 .
\]

(b) These values, obtained using Sage (a Mathematica-type software package), are

\[
\|f - h^*\|_{\infty} \approx 0.43948, \quad \|f - g^*\|_2 \approx 0.26822, \quad \|f - h^*\|_2 \approx 0.22946 .
\]

One should expect \( \|f - h^*\|_{\infty} \) to be larger than 0.279, but \( \|f - h^*\|_2 \leq \|f - g^*\|_2 \).

⋆24 (a) It suffices to show that each of \( \phi_j(x) = x^j, j = 0, 1, 2, 3 \) can be written as such a linear combination. Obviously this is so for \( \phi_0 \) and \( \phi_1 \). As for the others,

\[
x^2 =: \phi_2(x) = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x),
\]

and

\[
x^3 =: \phi_3(x) = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x).
\]

Because of this, every polynomial

\[
a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
= a_0 P_0(x) + a_1 P_1(x) + a_2 \left[ \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \right] + a_3 \left[ \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x) \right] \\
= \left( a_0 + \frac{a_2}{3} \right) P_0(x) + \left( a_1 + \frac{3a_3}{5} \right) P_1(x) + \frac{2a_2}{3} P_2(x) + \frac{2a_3}{5} P_3(x).
\]
(b) Observe, first, that whenever $P_i(x)P_j(x)$ is an odd function, then $\int_{-1}^{1} P_i(x)P_j(x) \, dx = 0$. Thus,

$$\langle P_0, P_1 \rangle = \langle P_0, P_3 \rangle = \langle P_1, P_2 \rangle = \langle P_2, P_3 \rangle = 0.$$  

Computing the other two pairings, we have

$$\langle P_0, P_2 \rangle = \frac{1}{2} \int_{-1}^{1} (3x^2 - 1) \, dx = \frac{1}{2} \left[ x^3 \right]_{-1}^{1} = \frac{1}{2} \left[ (1 - 1) - (-1 + 1) \right] = 0,$$

$$\langle P_1, P_3 \rangle = \frac{1}{2} \int_{-1}^{1} (5x^4 - 3x^2) \, dx = \frac{1}{2} \left[ x^5 - x^3 \right]_{-1}^{1} = 0.$$  

(c) It remains to evaluate all the inner products of the form $\langle P_k, P_k \rangle$:

$$\langle P_0, P_0 \rangle = \int_{-1}^{1} \, dx = 2,$$

$$\langle P_1, P_1 \rangle = \int_{-1}^{1} x^2 \, dx = \frac{1}{3} x^3 \bigg|_{-1}^{1} = \frac{2}{3},$$

$$\langle P_2, P_2 \rangle = \frac{1}{4} \int_{-1}^{1} (9x^4 - 6x^2 + 1) \, dx = \frac{1}{4} \left[ \frac{9}{5} x^5 - 2x^3 + x \right]_{-1}^{1} = \frac{2}{5},$$

$$\langle P_3, P_3 \rangle = \frac{1}{4} \int_{-1}^{1} (25x^6 - 30x^4 + 9x^2) \, dx = \frac{1}{4} \left[ \frac{25}{7} x^7 - 6x^5 + 3x^3 \right]_{-1}^{1} = \frac{2}{7}.$$  

Thus, using the language of MATLAB, the Gram matrix

$$G = \text{diag}(2/3, 2/5, 2/7).$$

\textbf{25} (a) For $m = 1, 2, \ldots, k$ we have

$$a_m = c_{-m} + c_m = \frac{1}{2} \left[ \int_{-1}^{1} f(x)e^{im\pi x} \, dx + \int_{-1}^{1} f(x)e^{-im\pi x} \, dx \right]$$

$$= \frac{1}{2} \int_{-1}^{1} f(x) \left( e^{im\pi x} + e^{-im\pi x} \right) \, dx = \int_{-1}^{1} f(x) \cos(m\pi x) \, dx.$$  

Handling the case $m = 0$ separately, we have

$$a_0 = 2c_0 = \int_{-1}^{1} f(x) \cdot 1 \, dx = \int_{-1}^{1} f(x) \cos(0 \cdot \pi x) \, dx.$$  

Turning to the $b_m$, $m = 1, \ldots, k$, we have

$$b_m = i(c_m - c_{-m}) = \frac{i}{2} \left[ \int_{-1}^{1} f(x)e^{-im\pi x} \, dx - \int_{-1}^{1} f(x)e^{im\pi x} \, dx \right]$$

$$= \frac{i}{2} \int_{-1}^{1} f(x) \left( e^{-im\pi x} - e^{im\pi x} \right) \, dx = \int_{-1}^{1} f(x) \sin(m\pi x) \, dx.$$  

(b) Here are my routines:

\textbf{PS13—Solutions}
function [a, b] = fourierCoeffs(f, k)
% function [a, b] = fourierCoeffs(f, k)
%
% This routine computes the classical Fourier series coefficients
% for f up to order k on the interval [-1, 1].
%
% INPUTS:
%  f   a function handle, pointing to the function to approximate
%  k   the number of sine terms to include
%
% OUTPUTS:
%  a   the coefficients of the cosine terms
%  b   the coefficients of the sine terms
a = quad(@(x) f(x), -1, 1);
b = [];
for m = 1:k
    a = [a; quad(@(x) f(x)*cos(m*pi*x), -1, 1)];
    b = [b; quad(@(x) f(x)*sin(m*pi*x), -1, 1)];
end
end

function y = fsEval(xs, a, b)
% function y = fsEval(xs, a, b)
%
% This routine computes the values of the truncated classical
% Fourier series on [-1, 1] with cosine coefficients in a and
% sine coefficients in b. The values correspond to the numbers
% (inputs) in the vector xs.
%
% INPUTS:
%  xs   A vector of numbers at which the truncated FS should be evaluated
%  a    The coefficients of the cosine terms
%  b    The coefficients of the sine terms
%
% OUTPUTS: A vector of values corresponding to those in xs
y = a(1)/2*ones(size(xs));
for j = 1:length(b)
    y = y + a(j+1)*cos(j*pi*xs) + b(j)*sin(j*pi*xs);
end
end
(c) I have plotted the truncated series for $k = 5$ (so 11 basis elements) in red, for $k = 9$ in green, and for $k = 25$ in yellow. It would seem that, as more basis functions are used, the truncated Fourier series converges to $f$ at most points in $[-1, 1]$, though noticeably not at $x = \pm 1$. The periodicity of these basis elements make it impossible to converge to this nonperiodic $f$ outside $[-1, 1]$. 