MATH 335: Numerical Analysis

Problem Set 13, Solutions

$\star$<u>23</u> (a) Lets define $\phi_0(x) = 1$, $\phi_1(x) = x$. Since we are integrating over an interval that is symmetric around zero, and ϕ_0, ϕ_1 are even, odd functions respectively, $\left\langle \phi_0, \phi_1 \right\rangle = 0$. Next, we have

$$\begin{aligned}
\left\langle \phi_0, \phi_0 \right\rangle &= \int_{-1}^{1} 1\, dx = 2 , \\
\left\langle \phi_1, \phi_1 \right\rangle &= \int_{-1}^{1} x^2\, dx = \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{2}{3} , \\
\left\langle f, \phi_0 \right\rangle &= \int_{-1}^{1} e^x\, dx = e^x \Big|_{-1}^{1} = e - e^{-1} \doteq 2.3504 , \\
\left\langle f, \phi_1 \right\rangle &= \int_{-1}^{1} x e^x\, dx = (x-1)e^x \Big|_{-1}^{1} = 2e^{-1} \doteq 0.73576 .
\end{aligned}$$

Thus, we have the matrix system

$$\begin{bmatrix} 2 & 0 \\ 0 & 2/3 \end{bmatrix} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} e - e^{-1} \\ 2e^{-1} \end{bmatrix} \qquad \Rightarrow \qquad \begin{aligned} b &= (e - e^{-1})/2 \doteq 1.1752;, \\ m &= 3e^{-1} = 1.1036 . \end{aligned}$$

(b) These values, obtained using Sage (a Mathematica-type software package), are

$$\|f - h^*\|_\infty \doteq 0.43948 , \quad \|f - g^*\|_2 \doteq 0.26822 , \quad \|f - h^*\|_2 \doteq 0.22946 .$$

One *should* expect $\|f - h^*\|_\infty$ to be larger than 0.279, but $\|f - h^*\|_2 \leq \|f - g^*\|_2$.

$\star$<u>24</u> (a) It suffices to show that each of $\phi_j(x) = x^j$, $j = 0, 1, 2, 3$ can be written as such a linear combination. Obviously this is so for ϕ_0 and ϕ_1. As for the others,

$$x^2 =: \phi_2(x) = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) ,$$

and

$$x^3 =: \phi_3(x) = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x) .$$

Because of this, every polynomial

$$\begin{aligned}
&a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
&= a_0 P_0(x) + a_1 P_1(x) + a_2 \left[\frac{1}{3} P_0(x) + \frac{2}{3} P_2(x) \right] + a_3 \left[\frac{3}{5} P_1(x) + \frac{2}{5} P_3(x) \right] \\
&= \left(a_0 + \frac{a_2}{3} \right) P_0(x) + \left(a_1 + \frac{3a_3}{5} \right) P_1(x) + \frac{2a_2}{3} P_2(x) + \frac{2a_3}{5} P_3(x) .
\end{aligned}$$

(b) Observe, first, that whenever $P_i(x)P_j(x)$ is an odd function, then $\int_{-1}^{1} P_i(x)P_j(x)\,dx = 0$. Thus,

$$\langle P_0, P_1\rangle = \langle P_0, P_3\rangle = \langle P_1, P_2\rangle = \langle P_2, P_3\rangle = 0\,.$$

Computing the other two pairings, we have

$$\begin{aligned}
\langle P_0, P_2\rangle &= \frac{1}{2}\int_{-1}^{1}(3x^2-1)\,dx = \frac{1}{2}(x^3-x)\Big|_{-1}^{1} = \frac{1}{2}[(1-1)-(-1+1)] = 0\,,\\
\langle P_1, P_3\rangle &= \frac{1}{2}\int_{-1}^{1}(5x^4-3x^2)\,dx = \frac{1}{2}(x^5-x^3)\Big|_{-1}^{1} = 0\,.
\end{aligned}$$

(c) It remains to evaluate all the inner products of the form $\langle P_k, P_k\rangle$:

$$\begin{aligned}
\langle P_0, P_0\rangle &= \int_{-1}^{1} dx = 2\,,\\
\langle P_1, P_1\rangle &= \int_{-1}^{1} x^2\,dx = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{2}{3}\,,\\
\langle P_2, P_2\rangle &= \frac{1}{4}\int_{-1}^{1}(9x^4-6x^2+1)\,dx = \frac{1}{4}\left(\frac{9}{5}x^5-2x^3+x\right)\Big|_{-1}^{1} = \frac{2}{5}\,,\\
\langle P_3, P_3\rangle &= \frac{1}{4}\int_{-1}^{1}(25x^6-30x^4+9x^2)\,dx = \frac{1}{4}\left(\frac{25}{7}x^7-6x^5+3x^3\right)\Big|_{-1}^{1} = \frac{2}{7}\,.
\end{aligned}$$

Thus, using the language of OCTAVE/MATLAB, the Gram matrix

```
> G = diag([2  2/3  2/5  2/7]).
```

★25 (a) For $m = 1, 2, \ldots, k$ we have

$$\begin{aligned}
a_m = c_{-m} + c_m &= \frac{1}{2}\left[\int_{-1}^{1} f(x)e^{im\pi x}\,dx + \int_{-1}^{1} f(x)e^{-im\pi x}\,dx\right]\\
&= \frac{1}{2}\int_{-1}^{1} f(x)\left(e^{im\pi x}+e^{-im\pi x}\right)dx = \int_{-1}^{1} f(x)\cos(m\pi x)\,dx\,.
\end{aligned}$$

Handling the case $m = 0$ separately, we have

$$a_0 = 2c_0 = \int_{-1}^{1} f(x)\cdot 1\,dx = \int_{-1}^{1} f(x)\cos(0\cdot\pi x)\,dx\,.$$

Turning to the b_m, $m = 1, \ldots, k$, we have

$$\begin{aligned}
b_m = i(c_m - c_{-m}) &= \frac{i}{2}\left[\int_{-1}^{1} f(x)e^{-im\pi x}\,dx - \int_{-1}^{1} f(x)e^{im\pi x}\,dx\right]\\
&= \frac{i}{2}\int_{-1}^{1} f(x)\left(e^{-im\pi x}-e^{im\pi x}\right)dx = \int_{-1}^{1} f(x)\sin(m\pi x)\,dx\,.
\end{aligned}$$

(b) Here are my routines:

```
function [a, b] = fourierCoeffs(f, k)
% function [a, b] = fourierCoeffs(f, k)
%
%   This routine computes the classical Fourier series coefficients
%   for f up to order k on the interval [-1, 1].
%
%   INPUTS:
%      f     a function handle, pointing to the function to approximate
%      k     the number of sine terms to include
%
%   OUTPUTS:
%      a     the coefficients of the cosine terms
%      b     the coefficients of the sine terms
  a = quad(@f, -1, 1);
  b = [];
  for m = 1:k
    a = [a; quad(@(x) f(x)*cos(m*pi*x), -1, 1)];
    b = [b; quad(@(x) f(x)*sin(m*pi*x), -1, 1)];
  end
end

function y = fsEval(xs, a, b)
% function y = fsEval(xs, a, b)
%
%   This routine computes the values of the truncated classical
%   Fourier series on [-1, 1] with cosine coefficients in a and
%   sine coefficients in b.  The values correspond to the numbers
%   (inputs) in the vector xs.
%
%   INPUTS:
%     xs     A vector of numbers at which the truncated FS should be evaluated
%      a     The coefficients of the cosine terms
%      b     The coefficients of the sine terms
%
%   OUTPUTS: A vector of values corresponding to those in xs
  y = a(1)/2*ones(size(xs));
  for j = 1:length(b)
    y = y + a(j+1)*cos(j*pi*xs) + b(j)*sin(j*pi*xs);
  end
end
```

(c) I have plotted the truncated series for $k = 5$ (so 11 basis elements) in red, for $k = 9$ in green, and for $k = 25$ in yellow. It would seem that, as more basis functions are used, the truncated fourier series converges to f at most points in $[-1, 1]$, though noticeably not at $x = \pm 1$. The periodicity of these basis elements make it impossible to converge to this nonperiodic f outside $[-1, 1]$.