MATH 335: Numerical Analysis

Problem Set 11, Solutions

★17 (a) I am using OCTAVE's `polyfit` command:

```
octave:3> f = @(x) x.^3 - x.^2 + 2*x - 7;
octave:5> x = [1 3 4.2 8];

octave:6> polyfit(x, f(x), 3)
ans =
   1.0000  -1.0000   2.0000  -7.0000
```

The function f itself is a 3rd-degree polynomial, and we used 4 points from its graph. There is only one polynomial of degree 3 which passes through the four points, so it should not be surprising that the coefficients returned by `polyfit()` match those in f itself.

(b)

```
octave:8> x = 1:10;
octave:9> polyfit(x, f(x), 9)
ans =
 Columns 1 through 6:
   2.4549e-17  -1.1930e-15   2.4864e-14  -2.9075e-13   2.0937e-12  -9.5808e-12

 Columns 7 through 10:
   1.0000e+00  -1.0000e+00   2.0000e+00  -7.0000e+00
```

We know, in fact, that there is only one polynomial of degree *at most 9* which passes through our 10 points. But f is such a polynomial, and hence is the only one—in theory, the one returned by the command. Indeed, when we look at the coefficients for the x^k terms, $k = 0, 1, 2$, or 3, they match those found in f. But there are other terms which the theory says ought to be zero and aren't. Perhaps the ill-conditioned nature of the Vandermonde matrix has less to do with the matrix iteslf and more do to with it just being a very sensitive process determing the polynomial.

5.4.2 (a) The cardinal functions are

$$L_1(x) = \left(\frac{x-2}{0-2}\right)\left(\frac{x-4}{0-4}\right)\left(\frac{x-6}{0-6}\right), \qquad L_3(x) = \left(\frac{x}{4}\right)\left(\frac{x-2}{4-2}\right)\left(\frac{x-6}{4-6}\right),$$

$$L_2(x) = \left(\frac{x}{2}\right)\left(\frac{x-4}{2-4}\right)\left(\frac{x-6}{2-6}\right), \qquad L_4(x) = \left(\frac{x}{6}\right)\left(\frac{x-2}{6-2}\right)\left(\frac{x-4}{6-4}\right).$$

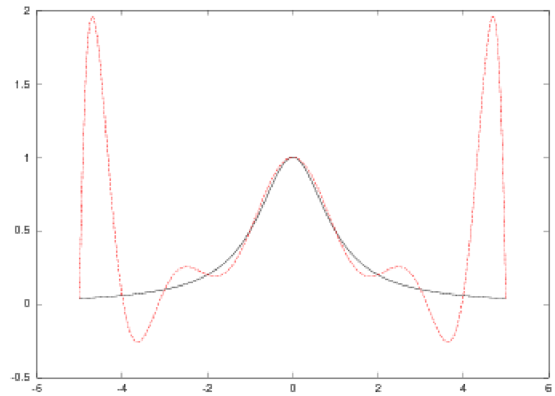
(b) The interpolating polynomial is

$$p(x) = L_1(x) - L_2(x) + 3\,L_3(x) + 4\,L_4(x)\ .$$

AP 5.4 (a) Working much as in Problem ⋆17, and treating those coefficients which are less than 10^{-4} as if they are zeros, I get

$$p_{10}(x) = 0.001267x^8 - 0.024412x^6 + 0.19738x^4 - 0.67421x^2 + 1 .$$

(b)
```
octave:14> c = polyfit(x, f(x), 10);
octave:17> x = [4.2 4.4 4.6 4.8];
octave:16> f(x) - polyval(c, x)
ans =
  -0.50680  -1.19110  -1.80072  -1.76279
```



Of course, even though these numbers may illustrate the principle, a picture is still worth a thousand words.

⋆18 (a) $\begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} = x_2 - x_1$

(b) $$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = \begin{vmatrix} x_2 & x_2^2 \\ x_3 & x_3^2 \end{vmatrix} - x_1 \begin{vmatrix} 1 & x_2^2 \\ 1 & x_3^2 \end{vmatrix} + x_1^2 \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} = x_2x_3^2 - x_3x_2^2 - x_1(x_3^2 - x_2^2) + x_1^2(x_3 - x_2)$$

$$= (x_3 - x_2)[x_2x_3 - x_1(x_3 + x_2) + x_1^2] = (x_3 - x_2)[x_3(x_2 - x_1) - x_1(x_2 - x_1)]$$

$$= (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)$$

(c) Optional.

(d) Suppose two points, say x_k and x_ℓ, are equal, then without loss of generality, we may assume that $k < \ell$ and the factor $(x_\ell - x_k)$, which is zero, appears in the product $\prod_{1\le j<i\le n}(x_i - x_j) = \det(\mathbf{V})$, making this determinant zero (and hence V singular).

Now suppose that no two points are equal. Then every factor $(x_i - x_j)$ in the determinant is nonzero, giving that the determinant is nonzero, and $\mathbf{V}$ is nonsingular.

5.3.7 We have $0.5 = x_1 < x_2 < \cdots < x_n = 1$, with each $x_{j+1} - x_j = h = 0.5/(n-1)$. The function $\ln x$ is as differentiable in $[0.5, 1]$ as we need it to be:

$$f(x) = \ln x \quad \Rightarrow \quad f^{(n)}(x) = (-1)^{n-1}x^{-n}\,(n-1)! \quad \Rightarrow \quad |f^{(n)}(x)| \le 2^n\,(n-1)!\,,$$

for all $x \in [0.5, 1]$. Thus, by our work in class, we have already that

$$|f(t) - p(t)| \le \frac{2^n(n-1)!}{4n}\,h^n = \frac{2^n(n-1)!}{4n} \cdot \frac{1}{2^n(n-1)^n} = \frac{(n-2)!}{4n(n-1)^{n-1}} .$$

If we can choose n large enough so that the expression at the far right is less than or equal to 10^{-3}, such n are sufficient. By trial-and-error we find $n \geq 6$ is sufficiently large.

5.3.8 (a) i. Here, of course, derivatives of f of all orders exist at every real x, with

$$f^{(n)}(x) \;=\; \begin{cases} (-1)^{(n-1)/2}\cos x\,, & n \text{ odd},\\ (-1)^{n/2}\sin x\,, & n \text{ even}. \end{cases}$$

ii. We have $|f^{(n)}(x)| \leq 1$ for all real $x \in [0,2]$. Let $p_n(t)$ be the interpolating polynomial of degree n. The formulas for $f(t) - p_n(t)$ developed in class must be adapted to the fact that we are using $(n+1)$ nodes $0 = x_0 < x_1 < \cdots < x_n = 2$. We have

$$|f(t) - p_n(t)| \;\leq\; \frac{|f^{(n+1)}(z)|}{(n+1)!}\prod_{j=0}^{n}|t - x_j| \;\leq\; \frac{1}{(n+1)!}\cdot\frac{1}{4}n!\left(\frac{2}{n}\right)^{n+1} \;=\; \frac{2^{n+1}}{4(n+1)n^{n+1}}\,.$$

iii. Since

$$|f(t) - p_n(t)| \leq \frac{1}{4(n+1)}\cdot\left(\frac{2}{n}\right)^{n+1},$$

and $(2/n) \to 0$ as $n \to \infty$, the error at any $t \in [0,2]$ is bounded by smaller and smaller numbers. This means the approximating polynomials $(p_n)_{n=1}^{\infty}$ converge *uniformly* to f.

(b) i. This time $f^{(n)}(x) = (-1)^n n!\,(x+1)^{-(n+1)}$. This n^{th} order derivative exists for $n = 1, 2, \ldots$ for all $x \in [0,2]$.

ii. For $x \in [0,2]$ we have

$$|f^{(n)}(x)| \;=\; \frac{n!}{(x+1)^{n+1}} \;\leq\; n!\,,$$

since the fraction $n!/(x+1)^{n+1}$ is made largest when $x = 0$. Thus,

$$|f(t) - p_n(t)| \;=\; \frac{|f^{(n+1)}(z)|}{(n+1)!}\prod_{j=0}^{n}|t - x_j| \;\leq\; \frac{(n+1)!}{(n+1)!}\cdot\frac{1}{4}n!\left(\frac{2}{n}\right)^{n+1} \;=\; \frac{2^{n-1}(n-1)!}{n^n}\,.$$

iii. It appears, numerically, that the sequence $\frac{2^{n-1}(n-1)!}{n^n} \to 0$ as $n \to \infty$. If this is so, then the approximating polynomials $(p_n)_{n=1}^{\infty}$ converge *uniformly* to f.

★19 (a) Here $f(x) = x^2 - (c+d)x + cd$, so $f'(x) = 2x - c - d$, which has its sole zero at $x = (c+d)/2$. Restricting our attention to the interval $[c,d]$ (which contains our critical number $(c+d)/2$), we have

$$|f(c)| \;=\; 0\,, \qquad |f(c)| \;=\; 0\,, \qquad \text{and} \qquad \left|f\left(\frac{c+d}{2}\right)\right| = \frac{1}{4}(d-c)^2\,,$$

showing we get the maximum of $|f|$ in $[c,d]$ at $(c+d)/2$.

(b) Since $t \in [x_1, x_n]$, there is some $k \in \{1, 2, \ldots, n-1\}$ for which $x_k \le t \le x_{k+1}$. Now $(t - x_k)$ and $(t - x_{k+1})$ are two of the factors in $\Psi(t)$, and by part (a),

$$|(t - x_k)(t - x_{k+1})| \;\le\; \frac{1}{4}(x_{k+1} - x_k)^2 \;=\; \frac{1}{4}h^2 \,.$$

Thus,

$$|\Psi(t)| \;=\; |t-x_1|\cdots|t-x_k|\,|t-x_{k+1}|\cdots|t-x_n| \;\le\; \frac{1}{4}h^2\,|t-x_1|\cdots|t-x_{k-1}|\,|t-x_{k+2}|\cdots|t-x_n| \,.$$

To get the rest of the bound we seek, we note that, in the *worst case* when, say, $t \in [x_1, x_2]$ (i.e., t is to the extreme left end of the interval), we then have

$$\begin{aligned}
|\Psi(t)| &= |t - x_1|\,|t - x_2|\,|t - x_3|\cdots|t - x_n| \\
&\le \frac{1}{4}h^2\,|t - x_3|\cdots|t - x_n| \\
&\le \frac{1}{4}h^2(2h)(3h)\cdots[(n-1)h] \\
&= \frac{1}{4}(n-1)!\,h^n \,.
\end{aligned}$$