17 (a) I am using Octave’s polyfit command:

```
octave:3> f = @(x) x.^3 - x.^2 + 2*x - 7;
octave:5> x = [1 3 4.2 8];

octave:6> polyfit(x, f(x), 3)
ans =
    1.0000 -1.0000  2.0000 -7.0000
```

The function \( f \) itself is a 3rd-degree polynomial, and we used 4 points from its graph. There is only one polynomial of degree 3 which passes through the four points, so it should not be surprising that the coefficients returned by \texttt{polyfit()} match those in \( f \) itself.

(b) \( \text{octave:8> x = 1:10;} \)

```
octave:9> polyfit(x, f(x), 9)
ans =
Columns 1 through 6:
    2.4549e-17 -1.1930e-15  2.4864e-14 -2.9075e-13  2.0937e-12 -9.5808e-12
Columns 7 through 10:
    1.0000e+00 -1.0000e+00  2.0000e+00 -7.0000e+00
```

We know, in fact, that there is only one polynomial of degree \emph{at most} 9 which passes through our 10 points. But \( f \) is such a polynomial, and hence is the only one—in theory, the one returned by the command. Indeed, when we look at the coefficients for the \( x^k \) terms, \( k = 0, 1, 2, \text{ or } 3 \), they match those found in \( f \). But there are other terms which the theory says ought to be zero and aren’t. Perhaps the ill-conditioned nature of the Vandermonde matrix has less to do with the matrix itself and more do to with it just being a very sensitive process determining the polynomial.

5.4.2 (a) The cardinal functions are

\[
L_1(x) = \frac{(x-2)(x-4)(x-6)}{0-2(0-4)(0-6)}, \quad L_3(x) = \frac{(x)(x-2)(x-6)}{4-2(4-6)},
\]

\[
L_2(x) = \frac{(x)(x-4)(x-6)}{2-4(2-6)}, \quad L_4(x) = \frac{(x)(x-2)(x-4)}{6-2(6-4)}.
\]

(b) The interpolating polynomial is

\[
p(x) = L_1(x) - L_2(x) + 3 L_3(x) + 4 L_4(x).
\]
AP 5.4  (a) Working much as in Problem \(\star 17\), and treating those coefficients which are less than \(10^{-4}\) as if they are zeros, I get
\[
p_{10}(x) = 0.001267x^8 - 0.024412x^6 + 0.19738x^4 - 0.67421x^2 + 1 .
\]

(b) octave:14> c = polyfit(x, f(x), 10);
octave:17> x = [4.2 4.4 4.6 4.8];
octave:16> f(x) - polyval(c, x)
ans =
     -0.50680  -1.19110  -1.80072  -1.76279
Of course, even though these numbers may illustrate the principle, a picture is still worth a thousand words.

\(\star 18\)  (a) \[
\begin{vmatrix}
 1 & x_1 \\
 1 & x_2
\end{vmatrix}
= x_2 - x_1
\]

(b) \[
\begin{vmatrix}
 1 & x_1 & x_1^2 \\
 1 & x_2 & x_2^2 \\
 1 & x_3 & x_3^2
\end{vmatrix}
= \begin{vmatrix}
 x_2 & x_1^2 - x_1 \\
 x_3 & x_2^2 - x_1 \\
 1 & x_3
\end{vmatrix}
+ x_1 \begin{vmatrix}
 1 & x_2 \\
 1 & x_3
\end{vmatrix}
= x_2x_3^2 - x_3x_2^2 - x_1(x_3^2 - x_2^2) + x_1^2(x_3 - x_2)
\]
\[
= (x_3 - x_2)[x_2x_3 - x_1(x_3 + x_2) + x_1^2] = (x_3 - x_2)[x_3(x_2 - x_1) - x_1(x_2 - x_1)]
\]
\[
= (x_3 - x_2)(x_2 - x_1)(x_3 - x_1)
\]

(c) Optional.

(d) Suppose two points, say \(x_k\) and \(x_\ell\), are equal, then without loss of generality, we may assume that \(k < \ell\) and the factor \((x_\ell - x_k)\), which is zero, appears in the product \(\prod_{1 \leq j < i \leq n}(x_i - x_j) = \det(V)\), making this determinant zero (and hence \(V\) singular).

Now suppose that no two points are equal. Then every factor \((x_i - x_j)\) in the determinant is nonzero, giving that the determinant is nonzero, and \(V\) is nonsingular.

5.3.7 We have \(0.5 = x_1 < x_2 < \cdots < x_n = 1\), with each \(x_{j+1} - x_j = h = 0.5/(n-1)\). The function \(\ln x\) is as differentiable in \([0.5, 1]\) as we need it to be:
\[
f(x) = \ln x \quad \Rightarrow \quad f^{(n)}(x) = (-1)^{n-1}x^{-n} (n-1)! \quad \Rightarrow \quad |f^{(n)}(x)| \leq 2^n (n-1)! ,
\]
for all \(x \in [0.5, 1]\). Thus, by our work in class, we have already that
\[
|f(t) - p(t)| \leq \frac{2^n(n-1)!}{4n} h^n = \frac{2^n(n-1)!}{4n} \cdot \frac{1}{2^n(n-1)^n} = \frac{(n-2)!}{4n(n-1)^{n-1}} .
\]
If we can choose \( n \) large enough so that the expression at the far right is less than or equal to \( 10^{-3} \), such \( n \) are sufficient. By trial-and-error we find \( n \geq 6 \) is sufficiently large.

5.3.8 (a)  
i. Here, of course, derivatives of \( f \) of all orders exist at every real \( x \), with

\[
f^{(n)}(x) = \begin{cases} 
(-1)^{(n-1)/2} \cos x, & n \text{ odd}, \\
(-1)^{n/2} \sin x, & n \text{ even}.
\end{cases}
\]

ii. We have \( |f^{(n)}(x)| \leq 1 \) for all real \( x \in [0, 2] \). Let \( p_n(t) \) be the interpolating polynomial of degree \( n \). The formulas for \( f(t) - p_n(t) \) developed in class must be adapted to the fact that we are using \((n + 1)\) nodes \( 0 = x_0 < x_1 < \cdots < x_n = 2 \). We have

\[
|f(t) - p_n(t)| \leq \frac{|f^{(n+1)}(z)|}{(n+1)!} \prod_{j=0}^{n} |t - x_j| \leq \frac{1}{(n+1)!} \cdot \frac{1}{4} n! \left( \frac{2}{n} \right)^{n+1} = \frac{2^{n+1}}{4(n+1)n^{n+1}}.
\]

iii. Since

\[
|f(t) - p_n(t)| \leq \frac{1}{4(n+1)} \left( \frac{2}{n} \right)^{n+1},
\]

and \( (2/n) \to 0 \) as \( n \to \infty \), the error at any \( t \in [0, 2] \) is bounded by smaller and smaller numbers. This means the approximating polynomials \( (p_n)^\infty_{n=1} \) converge uniformly to \( f \).

(b)  
i. This time \( f^{(n)}(x) = (-1)^n n! (x + 1)^{-(n+1)} \). This \( n \)th order derivative exists for \( n = 1, 2, \ldots \) for all \( x \in [0, 2] \).

ii. For \( x \in [0, 2] \) we have

\[
|f^{(n)}(x)| = \frac{n!}{(x + 1)^{n+1}} \leq n!,
\]

since the fraction \( n!/ (x + 1)^{n+1} \) is made largest when \( x = 0 \). Thus,

\[
|f(t) - p_n(t)| = \frac{|f^{(n+1)}(z)|}{(n+1)!} \prod_{j=0}^{n} |t - x_j| \leq \frac{1}{(n+1)!} \cdot \frac{1}{4} n! \left( \frac{2}{n} \right)^{n+1} = \frac{2^{n-1}(n-1)!}{n^n}.
\]

iii. It appears, numerically, that the sequence \( \frac{2^{n-1}(n-1)!}{n^n} \to 0 \) as \( n \to \infty \). If this is so, then the approximating polynomials \( (p_n)^\infty_{n=1} \) converge uniformly to \( f \).

★19 (a) Here \( f(x) = x^2 - (c + d)x + cd \), so \( f'(x) = 2x - c - d \), which has its sole zero at \( x = (c + d)/2 \). Restricting our attention to the interval \([c, d]\) (which contains our critical number \((c + d)/2\)), we have

\[
|f(c)| = 0, \quad |f(c)| = 0, \quad \text{and} \quad \left| f\left( \frac{c + d}{2} \right) \right| = \frac{1}{4} (d - c)^2,
\]

showing we get the maximum of \(|f|\) in \([c, d]\) at \((c + d)/2\).
(b) Since \( t \in [x_1, x_n] \), there is some \( k \in \{1, 2, \ldots, n - 1\} \) for which \( x_k \leq t \leq x_{k+1} \). Now \( (t - x_k) \) and \( (t - x_{k+1}) \) are two of the factors in \( \Psi(t) \), and by part (a),

\[
| (t - x_k)(t - x_{k+1}) | \leq \frac{1}{4} (x_{k+1} - x_k)^2 = \frac{1}{4} h^2 .
\]

Thus,

\[
| \Psi(t) | = |t - x_1| \cdots |t - x_k| |t - x_{k+1}| \cdots |t - x_n| \leq \frac{1}{4} h^2 |t - x_1| \cdots |t - x_{k-1}| |t - x_{k+2}| \cdots |t - x_n| .
\]

To get the rest of the bound we seek, we note that, in the worst case when, say, \( t \in [x_1, x_2] \) (i.e., \( t \) is to the extreme left end of the interval), we then have

\[
| \Psi(t) | = |t - x_1| |t - x_2| |t - x_3| \cdots |t - x_n|
\leq \frac{1}{4} h^2 |t - x_3| \cdots |t - x_n|
\leq \frac{1}{4} h^2 (2h)(3h) \cdots [(n - 1)h]
= \frac{1}{4} (n - 1)! h^n .
\]