MATH 335: Numerical Analysis

Problem Set 8, Solutions

4.3.6 The scale vector is $\mathbf{c} = (5, 9, 9, 8)$. We opt to exchange rows 1 and 3, since $\mathbf{r} = (2/5, 1/9, 2/3, 3/8)$. At this point our vector $\mathbf{d} = (3, 2, 1, 4)$, and after the first elimination step we have

$$\begin{bmatrix} 0 & 14/3 & 2 & -8/3 \\ 0 & 7/3 & 15/2 & 19/6 \\ 6 & 2 & -9 & -5 \\ 0 & -1 & 15/2 & -11/2 \end{bmatrix},$$

yielding a new $\mathbf{r} = (14/15, 7/27, 1/8)$, and prompting us to make another row exchange, so that $\mathbf{d} = (3, 1, 2, 4)$. The next elimination step yields

$$\begin{bmatrix} 0 & 14/3 & 2 & -8/3 \\ 0 & 0 & 13/2 & 9/2 \\ 6 & 2 & -9 & -5 \\ 0 & 0 & 111/14 & -85/14 \end{bmatrix}.$$

Now we have $\mathbf{r} = (13/18, 111/112)$, showing that another row exchange is called for, and $\mathbf{d} = (3, 1, 4, 2)$. Thus, our 3rd pivot row is row 4.

4.3.10 (a) Employing both methods, we have

```
octave:115> format long
octave:116> x1 = hilb(3) \ [1; 0; 0]
x1 =
    9.00000000000002
  -36.00000000000014
   30.00000000000014
octave:117> geSolve(hilb(3), [1; 0; 0])
ans =
    9.00000000000003
  -36.00000000000014
   30.00000000000013
```

showing very little discrepancy between the two.

(b) Here we have

```
octave:118> H2 = [1 .5 .333; .5 .333 .25; .333 .25 .2];
octave:119> x2 = H2 \ [1; 0; 0]
x2 =
```

```
 9.67065522227168
-39.50816462757323
 33.28356483938419
```

The relative error (under the vector norm $\|\cdot\|_2$) in the two solutions is

```
octave:124> norm(x2-x1)/norm(x1)
ans =  0.101674162309816
```

It is interesting to note that the relative error (under the matrix norm $\|\cdot\|_2$) is

```
octave:125> norm(H2 - hilb(3)) / norm(hilb(3))
ans =  2.36688811684222e-04
```

In class we derived that

$$\text{(relative error in solutions)} \;\le\; \text{cond}(\mathbf{A}) \cdot \text{(relative error in RH-side vectors)}.$$

We have no corresponding result that bounds the relative error in solutions by the relative error in *matrices*. Nevertheless, it is interesting to note that the two matrices have similar condition numbers

```
octave:120> cond(hilb(3))
ans =  524.056777586064
octave:121> cond(H2)
ans =  572.736615893086
```

and that the relative error in solutions is about 429 times (of the same order of magnitude as the above condition numbers) larger than the relative error in matrices.

CP 4.3.8 If we multiply through both sides of the equation by $x^2(x+1)^2$, we get

$$\begin{aligned} 4x^3 + 4x^2 + x - 1 &= A_1 x(x+1)^2 + A_2(x+1)^2 + A_3 x^2(x+1) + A_4 x^2 \\ &= (A_1 + A_3)x^3 + (2A_1 + A_2 + A_3 + A_4)x^2 + (A_1 + 2A_2)x + A_2\,, \end{aligned}$$

after the various like-terms have been gathered together. Since this equation must hold for all real x, the coefficients on both sides must be the same for each term. That is,

$$\begin{aligned} A_1 + A_3 &= 4 \\ 2A_1 + A_2 + A_3 + A_4 &= 4 \\ A_1 + 2A_2 &= 1 \\ A_2 &= -1 \end{aligned} \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \\ -1 \end{bmatrix}.$$

Solving this matrix equation, we get

$$A_1 = 3\,, \quad A_2 = -1\,, \quad A_3 = 1\,, \quad A_4 = -2\,.$$

$\star$11 Using OCTAVE, we get the following results

```
octave:1> A = [2 1 2; 1 2 3; 4 1 2];
octave:3> format rat
octave:4> [L, U, P] = lu(A)
L =
         1          0          0
       1/4          1          0
       1/2        2/7          1

U =
         4          1          2
         0        7/4        5/2
         0          0        2/7

P =
         0          0          1
         0          1          0
         1          0          0
```

So, we need to solve (for $\mathbf{B}$) the equation

$$\mathbf{AB} = \mathbf{I}, \qquad \text{or} \qquad \mathbf{LUB} = \mathbf{P}.$$

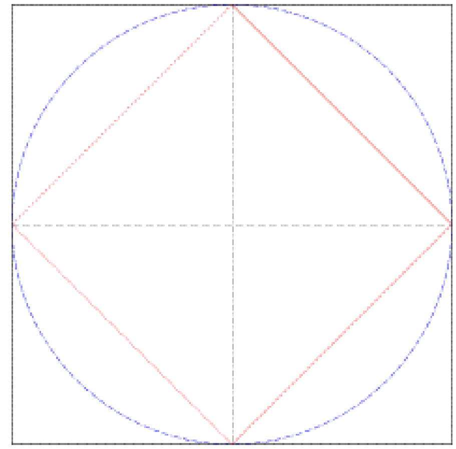
Let $\mathbf{y}_i$ be the solution of $\mathbf{Ly} = \mathbf{p}_i$, for $i = 1, 2, 3$, where $\mathbf{p}_i$ stands for the ith column of $\mathbf{P}$. Using forward substitution, we get

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ -2/7 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} 1 \\ -1/4 \\ -3/7 \end{bmatrix}.$$

Now we solve, using forward substitution, $\mathbf{Ux}_i = \mathbf{y}_i$ to get

$$\mathbf{x}_1 = \begin{bmatrix} -1/2 \\ -5 \\ 7/2 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1/2 \\ 2 \\ -3/2 \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^{-1} = \mathbf{B} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -5 & 2 & 2 \\ 7/2 & -1 & -3/2 \end{bmatrix}.$$

★12 (a) The three sets are pictured at right. The black square (which is the square region extending from bottom left corner at $(-1, -1)$ to top right at $(1, 1)$) corresponds to $p = \infty$, the red box to $p = 1$, and the blue circle to $p = 2$.



(b) From the previous part we learn that sets like $\left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_p = r\right\}$ are squares (with sides parallel to the coordinate axes) when $p = \infty$, circles when $p = 2$ and diamonds (squares with vertices on the coordinate axes) when $p = 1$. We might expect the following inequalities:

$$\begin{aligned} \|\mathbf{x}\|_\infty &:= \max\{|x_1|, |x_2|\} \\ &\leq \sqrt{x_1^2 + x_2^2} \quad =: \quad \|\mathbf{x}\|_2 \\ &= \sqrt{|x_1|^2 + |x_2|^2} \\ &\leq \sqrt{|x_1|^2 + 2|x_1||x_2| + |x_2|^2} \\ &= |x_1| + |x_2| \quad = \quad \|\mathbf{x}\|_1 \,. \end{aligned}$$

Thus, we may take $\lambda = \Gamma = C = 1$. Next, we note that the largest value of r for which the square $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty = r\}$ lies inside the circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2 = 1\}$ is $r = \sqrt{2}/2$; the largest r for which the square $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_\infty = r\}$ lies inside the diamond $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_1 = 1\}$ is $r = 1/2$; and the largest r for which the circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_1 = r\}$ lies inside the diamond $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_1 = 1\}$ is $r = 1/\sqrt{2}$. Thus, we may take $\gamma = \sqrt{2}/2$, $\Lambda = 2$, and $c = 1/\sqrt{2}$. That is,

$$\frac{1}{\sqrt{2}}\|\mathbf{v}\|_1 \leq \|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_1 \,, \qquad \frac{1}{\sqrt{2}}\|\mathbf{v}\|_2 \leq \|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_2 \,,$$

and

$$\|\mathbf{v}\|_\infty \leq \|\mathbf{v}\|_1 \leq 2\|\mathbf{v}\|_\infty \,, \qquad \text{for all } \mathbf{v} \in \mathbb{R}^2 \,.$$

★13 (a) Fix an n-by-n matrix $\mathbf{A}$. We note that the first inequality holds trivially if $\mathbf{w}$ is the zero vector. So let $w \in \mathbb{R}^n$ be nonzero, and define $\mathbf{u} = \mathbf{w}/\|\mathbf{w}\|$. Since $\|\mathbf{u}\| = 1$, we have

$$\|\mathbf{A}\mathbf{u}\| \leq \max\{\|\mathbf{A}\mathbf{v}\| : \|\mathbf{v}\| = 1\} =: \|\mathbf{A}\| \,.$$

But, employing the homogeneity of the vector norm,

$$||\mathbf{Au}|| \;=\; \left\|\mathbf{A}\frac{\mathbf{w}}{||\mathbf{w}||}\right\| \;=\; \frac{1}{||\mathbf{w}||}||\mathbf{Aw}||\;,$$

and thus we have

$$\frac{1}{||\mathbf{w}||}||\mathbf{Aw}|| \;\le\; ||\mathbf{A}|| \qquad \Rightarrow \qquad ||\mathbf{Aw}|| \;\le\; ||\mathbf{A}||||\mathbf{w}||\;.$$

For the 2nd inequality, fix the matrices $\mathbf{A}$, $\mathbf{B}$, and let $\mathbf{u} \in \mathbb{R}^n$ satisfy $||\mathbf{u}|| = 1$. Then, since $\mathbf{Bu} \in \mathbb{R}^n$, we have

$$||(\mathbf{AB})\mathbf{u}|| \;=\; ||\mathbf{A}(\mathbf{Bu})|| \;\le\; ||\mathbf{A}||\,||\mathbf{Bu}|| \;\le\; ||\mathbf{A}||\,||\mathbf{B}||\,||\mathbf{u}|| \;=\; ||\mathbf{A}||\,||\mathbf{B}||\;.$$

Since this inequality holds for all choices of $\mathbf{u} \in \mathbb{R}^n$ with $||\mathbf{u}|| = 1$, the maximum possible value of $||(\mathbf{AB})\mathbf{u}||$ over all such $\mathbf{u}$ is also bounded above by $||\mathbf{A}||\,||\mathbf{B}||$. That is, $||\mathbf{AB}|| \le ||\mathbf{A}||\,||\mathbf{B}||$.

(b) Noting that $\mathbf{I}^2 = \mathbf{I}$, we have

$$||\mathbf{I}|| \;=\; ||\mathbf{I}^2|| \;\le\; ||\mathbf{I}||\,||\mathbf{I}||\;,$$

where the inequality is just the 2nd inequality of part (a). Dividing both sides by $||\mathbf{I}||$, we get $1 \le ||\mathbf{I}||$. Now let $\mathbf{A}$ be any nonsingular n-by-n matrix. We have

$$1 \;\le\; ||\mathbf{I}|| \;=\; ||\mathbf{AA}^{-1}|| \;\le\; ||\mathbf{A}||\,||\mathbf{A}^{-1}|| \;=:\; \mathrm{cond}(\mathbf{A})\;.$$

(c) Since $||\cdot||$ is a submultiplicative matrix norm, we have

$$\begin{aligned} ||\mathbf{A}^2|| &\le ||\mathbf{A}||\,||\mathbf{A}|| = ||\mathbf{A}||^2\,, \\ ||\mathbf{A}^3|| &\le ||\mathbf{A}^2||\,||\mathbf{A}|| = ||\mathbf{A}||^3\,, \\ &\;\vdots \\ ||\mathbf{A}^k|| &\le ||\mathbf{A}||^k\,, \\ &\;\vdots \end{aligned}$$

Thus, if $||\mathbf{A}|| = \gamma < 1$, then

$$||\mathbf{A}^k - \mathbf{0}|| \;=\; ||\mathbf{A}^k|| \;\le\; ||\mathbf{A}||^k \;=\; \gamma^k \;\to\; 0\,, \qquad \text{as } k \to \infty\;.$$

(d) For the given $\mathbf{A}$, we have $||\mathbf{A}||_1 = 2.5$, $||\mathbf{A}||_2 \doteq 2.118$, and $||\mathbf{A}||_\infty = 2.5$. Nevertheless,

$$\mathbf{A}^2 = \begin{bmatrix} 2^{-2} & 2 \\ 0 & 2^{-2} \end{bmatrix},\;\; \mathbf{A}^3 = \begin{bmatrix} 2^{-3} & 3/2 \\ 0 & 2^{-3} \end{bmatrix},\;\; \mathbf{A}^4 = \begin{bmatrix} 2^{-4} & 4/2^2 \\ 0 & 2^{-3} \end{bmatrix},\ldots,\mathbf{A}^k = \begin{bmatrix} 2^{-k} & k/2^{k-2} \\ 0 & 2^{-k} \end{bmatrix},\cdots,$$

which implies $||\mathbf{A}^k - \mathbf{0}||_1 = ||\mathbf{A}^k||_1 = (1+4k)/2^{-k} \to 0$, as $k \to \infty$.