MATH 335: Numerical Analysis

Problem Set 4, Solutions

★3 (a) This equation follows from taking 4 times equation (**??**) and subtracting (**??**), then solving for $f'(x)$. One also must use the relationships $d_4 = -c_4/4$ and $d_6 = -5c_6/16$.

(b) Multiplying equation (**??**) by 16 and subtracting (**??**), we get

$$15f'(x) \; = \; 16D_{3,2} - D_{2,2} - \frac{3}{4}d_6h^6 - \cdots ,$$

or

$$\begin{aligned} f'(x) \; &= \; \frac{16}{15}D_{3,2} - \frac{1}{15}D_{2,2} - \frac{1}{20}d_6h^6 - \cdots \\ &= \; D_{3,3} + \mathcal{O}(h^6) \, . \end{aligned}$$

9.2.10 Let us denote $\int_a^b f(x)\,dx$ by J. Then we have

$$J \; = \; I(h) + K_1h + K_2h^3 + K_3h^5 + \cdots \tag{1}$$

$$J \; = \; I(h/3) + K_1\frac{h}{3} + K_2\frac{h^3}{27} + K_3\frac{h^5}{243} + \cdots \tag{2}$$

$$J \; = \; I(h/9) + K_1\frac{h}{9} + K_2\frac{h^3}{729} + K_3\frac{h^5}{59049} + \cdots \tag{3}$$

Multiplying (2) by 3 and subtracting (1) we get (after solving again for J)

$$J \; = \; \frac{3}{2}I(h/3) - \frac{1}{2}I(h) + c_2h^3 + c_3h^5 + \cdots , \tag{4}$$

where $c_2 = (-4/9)K_2$ and $c_3 = (-40/81)K_3$. Multiplying (3) by 3 and subtracting (2) gives

$$J \; = \; \frac{3}{2}I(h/9) - \frac{1}{2}I(h/3) + \frac{1}{27}c_2h^3 + \frac{1}{243}c_3h^5 + \cdots . \tag{5}$$

Equations (4) and (5) show that $[3I(h/3) - I(h)]/2$ and $[3I(h/9) - I(h/3)]/2$ both yield $\mathcal{O}(h^3)$ approximations to J. But we can do even better by multiplying (5) by 27, subtracting (4), and then solving for J to get

$$J \; = \; \frac{81}{52}I(h/9) - \frac{15}{26}I(h/3) + \frac{1}{52}I(h) + \mathcal{O}(h^5) \, .$$

Thus, $J \approx (81/52)(2.9795) - (15/26)(2.9263) + (2.3965)/(52) \doteq 2.9990$.

CP 9.2.2 Modification of the book's routine doesn't entail much—mostly commenting out the lines that do the printing (unless you still want it to be printed), and indicating that `D` is to be returned. Something like

```
function D = derive(f,h,a,n)
% Approximate the derivative of a function at x=a
%disp('       Derivative table')
%disp('_______________________________________________________')
%disp(' i   h            Di,1        Di,2        Di,3      ...  ')
%disp('_______________________________________________________')
D(1,1)=(feval(f,a+h)-feval(f,a-h))/(2*h);
%fprintf('%2.0f %8.4f %12.4f\n',1,h,D(1,1));
for i=1:n-1
   h=h/2;
   D(i+1,1)=(feval(f,a+h)-feval(f,a-h))/(2*h);
%  fprintf('%2.0f %8.4f %12.4f',i+1,h,D(i+1,1));
   for k=1:i
      D(i+1,k+1)=D(i+1,k)+(D(i+1,k)-D(i,k))/((4^k)-1);
%     fprintf('%12.4f',D(i+1,k+1));
   end
%  fprintf('\n');
end
```

(a) With format set to short, the output looks like

```
34.96000    0.00000    0.00000    0.00000
34.87000   34.84000    0.00000    0.00000
34.84750   34.84000   34.84000    0.00000
34.84187   34.84000   34.84000   34.84000
```

Here, as below, the best approximation to the derivative appears in the bottom of the far-right column.

(b)

```
 0.77850   0.00000   0.00000   0.00000
 0.78917   0.79273   0.00000   0.00000
 0.79177   0.79263   0.79262   0.00000
 0.79241   0.79262   0.79262   0.79262
```

(c)

```
-4.20387   0.00000   0.00000   0.00000
-4.20821  -4.20966   0.00000   0.00000
-4.20930  -4.20966  -4.20966   0.00000
-4.20957  -4.20966  -4.20966  -4.20966
```

(d)

```
 1.69655   0.00000   0.00000   0.00000
```

```
1.70446   1.70709   0.00000   0.00000
1.70644   1.70711   1.70711   0.00000
1.70694   1.70711   1.70711   1.70711
```

(e)

```
28.66820    0.00000    0.00000    0.00000
28.40297   28.31456    0.00000    0.00000
28.33683   28.31479   28.31480    0.00000
28.32031   28.31480   28.31480   28.31480
```

AP 9.3 We are all going to guess at values here. Here are some possible values for the amounts at times surrounding $t = 4$:

time	amount	time	amount
0	800	16	250
4	755	12	480
6	730	10	590
7	700	9	650

These yield the Richardson table (with values approximating the rate of change at $t = 8$):

i	h	$D_{i,1} = \frac{f(x+h)-f(x-h)}{2h}$	$D_{i,2}$	$D_{i,3}$	$D_{i,4}$
1	8	-34.375			
2	4	-34.375	-34.375		
3	2	-35.000	-35.208	-35.264	
4	1	-25.000	-21.667	-20.764	-21.452

3.1.8 The function has a double root, so its derivative $f'(x) = 4x^3 - 25.8x^2 - 71.02x + 464.4$ should have a single root at the same location. Using graphs of f, f', it appears they have a common zero in the interval $[4, 4.5]$. Calling the routine `bisect.m` (and giving it f' instead of f), we get

```
octave:382> function y = fprime(x)
>    y = 4 * x.^3 - 25.8 * x.^2 - 71.02 * x + 464.4;
> endfunction

octave:383> bisect(@fprime, 4, 4.5, .000001, 1000)
__________________________________________________________________________
 iter    a            b            c            f(c)        |b-a|/2
__________________________________________________________________________
```

```
 0      4.0000      4.5000      4.250000    3.615000    0.250000
 1      4.2500      4.5000      4.375000   -5.179688    0.125000

 9      4.2998      4.3008      4.300293   -0.020804    0.000488
10      4.2998      4.3003      4.300049   -0.003468    0.000244
11      4.2998      4.3000      4.299927    0.005202    0.000122

15      4.3000      4.3000      4.299995    0.000325    0.000008
16      4.3000      4.3000      4.299999    0.000054    0.000004
17      4.3000      4.3000      4.300001   -0.000081    0.000002
```

So, this zero of f is approximately 4.3. Similarly,

```
octave:382> function y = f(x)
>    y = x.^4 - 8.6 * x.^3 - 35.51 * x.^2 + 464.4 * x - 998.46;
> endfunction

octave:396> bisect(@f, -7.5, -7, 0.000001, 1000)
_____________________________________________________________________
 iter    a            b            c            f(c)       |b-a|/2
_____________________________________________________________________


10     -7.3486     -7.3481     -7.348389   -0.160641    0.000244
11     -7.3486     -7.3484     -7.348511    0.082787    0.000122

15     -7.3485     -7.3485     -7.348473    0.006714    0.000008
16     -7.3485     -7.3485     -7.348469   -0.000893    0.000004
17     -7.3485     -7.3485     -7.348471    0.002911    0.000002
```

3.1.13 (a) We have that f is continuous on $[1,2]$ with $f(1) = 1 > 0$ and $f(2) = -\ln 2 < 0$. By the IVT there is *at least* one zero of f in $[1,2]$. To see that there is *at most* one, we note that $f'(x) = 2(x-2) - 1/x$ and $f''(x) = 2 + 1/x^2$. The latter shows that on $[1,2]$, $f''(x)$ is always of the same sign, so $f'(x)$ increases on $[1,2]$ reaching its maximum value (on that interval) at $f'(2) = -1/2$. Thus, $f(x)$ decreases on $[1,2]$ and can have at most one zero there.

(b) As we know the root lies between 1 and 2, 6 decimal digit accuracy would be accurate to with 0.000005.

```
octave:382> function y = f(x)
>   y = (x - 2).^2 - log(x);
> endfunction

octave:399> bisect(@f, 1, 2, 0.000005, 1000)
______________________________________________________________________
 iter     a            b            c            f(c)       |b-a|/2
______________________________________________________________________

 0      1.0000      2.0000      1.500000   -0.155465    0.500000
 1      1.0000      1.5000      1.250000    0.339356    0.250000

12      1.4124      1.4126      1.412476   -0.000159    0.000122
13      1.4124      1.4125      1.412415   -0.000044    0.000061
14      1.4124      1.4124      1.412384    0.000013    0.000031
15      1.4124      1.4124      1.412399   -0.000015    0.000015
16      1.4124      1.4124      1.412392   -0.000001    0.000008
```

(c) According to the formula on p. 51, we must solve

$$\frac{2-1}{2^{n+1}} \le \frac{1}{10^4} \qquad \Rightarrow \qquad n \ge 13 \, .$$