Solutions to PS #3

★1 (a) Using 3-digit arithmetic, we have "solutions"

$$\begin{aligned} x &= \frac{1}{2}\left(12.4 \pm \sqrt{(12.4)^2 - (4)(0.494)}\right) = \frac{1}{2}\left(12.4 \pm \sqrt{154 - 1.98}\right) \\ &= \frac{1}{2}\left(12.4 \pm \sqrt{152}\right) = \frac{1}{2}(12.4 \pm 12.3) = 0.05\,,\ \ 12.4\,. \end{aligned}$$

(b) The relative error is $|(0.0399675 - 0.5)/0.0399675| \doteq 0.25$ This is less than 5×10^{-k} for $k = 1$ but not $k = 2$, the approximation has 1 significant digit. The reason for the loss of significance is that, with these coefficients, $\sqrt{b^2 - 4ac} \approx |b|$, so the numerator subtracts to nearly zero for one of the roots.

(c) We have

$$r_{1,2} = \frac{1}{2a}(-b \pm \sqrt{b^2 - 4ac})\,.$$

Thus

$$r_1 r_2 = \left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) = \frac{1}{4a^2}[b^2 - (b^2 - 4ac)] = \frac{c}{a}\,.$$

(d) Here's my routine. The lines beginning with percent symbols are comments. When such lines appear immediately after the function declaration, they serve a special role: at the OCTAVE prompt, if you type `help quadFormula`, these lines will be printed to the screen.

```
function roots = quadFormula(coeffs)
% function roots = quadFormula(coeffs)
%
% written by TLS for Problem Set 3, MATH 335, Feb. 10, 2009
%
% Input coeffs should have length three  [a b c]
%   a = coeff. of x^2 term
%   b = coeff. of x term
%   c = constant term
%
% No special care was taken to make this routine work when
% the discriminant (b^2 - 4ac) is non-positive.

  if (length(coeffs) != 3)
    disp('Input vector is not of the appropriate length; it should be 3.')
```

```
      break
    else
      a = coeffs(1);
      b = coeffs(2);
      c = coeffs(3);
    end

    if (b < 0)
      roots = (-b + sqrt(b^2 - 4*a*c)) / (2 * a);
  roots = [c/(a*roots) roots];
    else
      roots = (-b - sqrt(b^2 - 4*a*c)) / (2 * a);
  roots = [roots c/(a*roots)];
    end
  end
```

(e) Repeating steps like those in part (a), we get a repeated root $x = 6.2$. Since there is no root with larger absolute value, the method used in part (d) cannot help. In the previous instance, loss of significance happened because $\sqrt{b^2 - 4ac} \approx |b|$. This time it happens because the quantity under the radical subtracts to near 0.

<u>2.2.5</u> We have

$$\begin{aligned}
\frac{X_1X_2 - x_1x_2}{X_1X_2} &= \frac{X_1X_2 - X_1x_2 + X_1x_2 - x_1x_2}{X_1X_2} = X_1\frac{X_2 - x_2}{X_1X_2} + x_2\frac{X_1 - x_1}{X_1X_2} \\
&= \frac{X_2 - x_2}{X_2} + \frac{x_2}{X_2} \cdot \frac{X_1 - x_1}{X_1} = \frac{\varepsilon_2}{X_2} + \frac{x_2}{X_2} \cdot \frac{\varepsilon_1}{X_1} \\
&\approx \frac{\varepsilon_2}{X_2} + \frac{\varepsilon_1}{X_1},
\end{aligned}$$

since $x_1 \approx X_1$.

<u>★2</u> (a) We assume there is some neighborhood surrounding x—say, $[x - \alpha, x + \beta]$—on which f is 3-times continuously differentiable. We also assume that h is small enough that both $(x - h)$ and $(x + h)$ lay inside the interval $[x - \alpha, x + \beta]$.

(b) We have already assumed ξ_1, ξ_2 lay inside the interval $[x - \alpha, x + \alpha]$ on which f''' is continuous. Clearly $[f'''(\xi_1) + f'''(\xi_2)]/2$ is a number between $f(\xi_1)$ and $f(\xi_2)$, so by the Intermediate Value Theorem there is a number ξ between ξ_1

and ξ_2 at which $f(\xi) = [f'''(\xi_1) + f'''(\xi_2)]/2$. Thus, the term

$$\frac{h^2}{12}[f'''(\xi_1) + f'''(\xi_2)] \quad \text{may be written as} \quad \frac{h^2}{6}f'''(\xi) \ .$$

9.1.3 (a) We write the Taylor series for $f(x+2h)$, $f(x-2h)$ about x:

$$\begin{aligned} f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(\xi_1) \\ f(x-2h) &= f(x) - 2hf'(x) + \frac{4h^2}{2}f''(\xi_2) \ , \end{aligned}$$

and then subtract these equations to get

$$f(x+2h) - f(x-2h) = 4hf'(x) + 2h^2[f''(\xi_1) - f''(\xi_2)] \ .$$

Solving for $f'(x)$, we get

$$\begin{aligned} f'(x) &= \frac{1}{4h}[f(x+2h) - f(x-2h)] - \frac{h}{2}[f''(\xi_1) - f''(\xi_2)] \\ &= \frac{1}{4h}[f(x+2h) - f(x-2h)] + O(h) \ . \end{aligned}$$

(b) We write expansions for $f(x+h)$ and $f(x+2h)$:

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(\xi_1) \\ f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(\xi_2) \end{aligned}$$

Multiplying the top equation by 4 and then subtracting gives

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + 2h^2[f''(\xi_1) - f''(\xi_2)] \ .$$

Solving for $f'(x)$, we get

$$\begin{aligned} f'(x) &= \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)] - h[f''(\xi_1) - f''(\xi_2)] \\ &= \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)] + O(h) \ . \end{aligned}$$

9.1.8–9 We do Taylor expansions of f at points h, $2h$ and $3h$ away from x:

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f^{(3)}(x) + \frac{h^4}{24}f^{(4)}(\xi_1) \ , \\ f(x+2h) &= f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{6}f^{(3)}(x) + \frac{16h^4}{24}f^{(4)}(\xi_2) \ , \\ f(x+3h) &= f(x) + 3hf'(x) + \frac{9h^2}{2}f''(x) + \frac{27h^3}{6}f^{(3)}(x) + \frac{81h^4}{24}f^{(4)}(\xi_3) \ , \end{aligned}$$

where ξ_1 is between x and $x+h$, ξ_2 is between x and $x+2h$, and ξ_3 is between x and $x+3h$. Multiplying these 3 equations through by 18, (-9) and 2 respectively, and adding the result, we get

$$\begin{aligned}
&18f(x+h) - 9f(x+2h) + 2f(x+3h) \\
&\quad = 11f(x) + 6hf'(x) + (9-18+9)h^2f''(x) + (3-12+9)h^3f^{(3)}(x) \\
&\qquad + \frac{1}{4}h^4\left[3f^{(4)}(\xi_1) - 24f^{(4)}(\xi_2) + 27f^{(4)}(\xi_3)\right] \\
&\quad = 11f(x) + 6hf'(x) + \frac{1}{4}h^4\left[3f^{(4)}(\xi_1) - 24f^{(4)}(\xi_2) + 27f^{(4)}(\xi_3)\right] .
\end{aligned}$$

Solving for $f'(x)$, we get

$$\begin{aligned}
f'(x) &= \frac{1}{6h}\left[18f(x+h) - 9f(x+2h) + 2f(x+3h) - 11f(x)\right] \\
&\quad - \frac{1}{8}h^3\left[f^{(4)}(\xi_1) - 8f^{(4)}(\xi_2) + 9f^{(4)}(\xi_3)\right] \\
&= \frac{1}{6h}\left[18f(x+h) - 9f(x+2h) + 2f(x+3h) - 11f(x)\right] + O(h^3) .
\end{aligned}$$

CP 1.3.2 (a) Repeating work like that of Exercise 1.3.21(b), we have

$$\cos t = 1 - \frac{1}{2!}t^2 + \cdots + \frac{(-1)^n}{(2n)!}t^{2n} + \frac{(-1)^{n+1}\cos\xi}{(2n+2)!}t^{2n+2} ,$$

which gives us that

$$\frac{1-\cos t}{t} = \underbrace{\left(\frac{1}{2!}t - \frac{1}{4!}t^3 + \cdots + \frac{(-1)^{n-1}}{(2n)!}t^{2n-1}\right)}_{\tau(t)} + \underbrace{(-1)^n\frac{\cos\xi}{(2n+2)!}t^{2n+1}}_{\gamma(t)} .$$

Thus,

$$\begin{aligned}
\int_0^x \frac{1-\cos t}{t}\,dt &= \int_0^x \tau(t)\,dt + \int_0^x \gamma(t)\,dt \\
&= \left.\frac{t^2}{2\cdot 2!} - \frac{t^4}{4\cdot 4!} + \cdots + (-1)^{n-1}\frac{t^{2n}}{(2n)(2n)!}\right|_0^x + \int_0^x \gamma(t)\,dt \\
&= T(x) + R(x) ,
\end{aligned}$$

where

$$T(x) := \frac{x^2}{2\cdot 2!} - \frac{x^4}{4\cdot 4!} + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)(2n)!} ,$$

and, assuming $x > 0$,

$$|R(x)| = \left|\int_0^x \gamma(t)\,dt\right| \le \int_0^x |\gamma(t)|\,dt = \int_0^x \frac{|\cos\xi|}{(2n+2)!}t^{2n+1}\,dt \le \frac{x^{2n+2}}{(2n+2)(2n+2)!} .$$

(b) Here is my *Cin*( ) function, which makes use of a pre-written `horner()` routine. The first thing it does is determine n large enough so that, for a user-specified ε (i.e., the value of *tol*),

$$\frac{|x|^{2n+2}}{(2n+2)(2n+2)!} < \varepsilon \qquad \Rightarrow \qquad |x| < [(2n+2)(2n+2)!\varepsilon]^{1/(2n+2)} .$$

```
function [ca, n] = cin(x, tol)
% function [ca, n] = cin(x, tol)
%
% Computes the "cosine integral" at x to within specified tolerance
% Output:  <ca>  is approximate value of the cosine integral
%          <n>   is the degree of the Taylor polynomial
%                whose value at x is used to compute <ca>.
  n = 0;
  while ( abs(x) > ((2*n+2)*gamma(2*n+3)*tol)^(1/(2*n+2)) )
    n = n + 1;
  end

  a = 0;
  for j = 1:n
    a = [a [0 (-1)^(j-1)/(2*j*gamma(2*j+1))]];
  end

  ca = horner(x, a);     % horner(0, a) will be zero
  n = 2*n;
end
```