Solutions to PS #1

1.7 One might do this by first demonstrating that, for $n \geq 1$,

$$f^{(n)}(x) \;=\; -(n-1)!(1-x)^{-n} \qquad \Rightarrow \qquad f^{(n)}(0) \;=\; -(n-1)!\,.$$

Another, perhaps slicker, approach might be to show that $f'(x) = -1/(1-x)$, and then employ the fact that

$$\frac{1}{1-x} \;=\; 1 + x + x^2 + \cdots + x^n + \cdots\,, \qquad \text{for } |x| < 1;$$

That is, we have the MacLaurin series for $-f'(x)$. Integrating term-by-term and using the fact that $f(0) = 0$, we get

$$f(x) = -(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots + \frac{1}{n}x^n + \cdots)\,, \qquad \text{for } |x| < 1\,.$$

1.9 Using knowledge of the derivatives of $\cos(\cdot)$ along with the facts that $\cos(\pi/3) = 1/2$ while $\sin(\pi/3) = \sqrt{3}/2$, we have the full series

$$\begin{aligned}
\cos x \;&=\; \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2\cdot 3!}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{2\cdot 4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots \\
&=\; \frac{1}{2}\left[1 - \frac{1}{2!}\left(x - \frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{3}\right)^4 + \cdots + (-1)^n\frac{1}{(2n)!}\left(x - \frac{\pi}{3}\right)^{2n} + \cdots\right] \\
&\qquad - \frac{\sqrt{3}}{2}\left[\left(x - \frac{\pi}{3}\right) - \frac{1}{3!}\left(x - \frac{\pi}{3}\right)^3 + \cdots + (-1)^{n+1}\frac{1}{(2n+1)!}\left(x - \frac{\pi}{3}\right)^{2n+1} + \cdots\right]\,,
\end{aligned}$$

for each real number x.

1.14 For $n > 0$, we have in general, $f^{(n)}(x) = (-1)^n n! x^{-(n+1)}$. Thus, employing Taylor's theorem with remainder, we have $f(x) = T_5(x) + R_6(x)$, where, for $x > 0$,

$$\begin{aligned}
T_5(x) \;&=\; \frac{4}{3} - \left(\frac{4}{3}\right)^2\left(x - \frac{3}{4}\right) + \left(\frac{4}{3}\right)^3\left(x - \frac{3}{4}\right)^2 - \left(\frac{4}{3}\right)^4\left(x - \frac{3}{4}\right)^3 + \left(\frac{4}{3}\right)^5\left(x - \frac{3}{4}\right)^4 - \left(\frac{4}{3}\right)^6\left(x - \frac{3}{4}\right)^5\,, \\
R_6(x) \;&=\; \xi^{-7}\left(x - \frac{3}{4}\right)^6\,,
\end{aligned}$$

with ξ between x and $3/4$. For $x \in [1/2, 1]$, $|f(x) - T_5(x)| = |R_6(x)| \leq (1/2)^{-7} \cdot (1/4)^6 = 1/32$.

1.15 If n terms of the Taylor series expansion of $f(x) = 1/x$ about $x_0 = 3/4$ are kept, then the remainder term will be

$$R_{n+1}(x) \;=\; (-1)^{n+1}\xi^{-(n+2)}\left(x - \frac{3}{4}\right)^{(n+1)}\,.$$

So, for $x \in [1/2, 1]$,

$$|R_{n+1}(x)| \;\le\; (1/2)^{-(n+2)} \cdot (1/4)^{(n+1)} \;=\; \left(\frac{1}{2}\right)^n .$$

To guarantee error less than ε, we choose n large enough so that

$$\frac{1}{2^n} \;<\; \varepsilon \qquad \Rightarrow \qquad n \;>\; -\frac{\log \varepsilon}{\log 2} .$$

For $\varepsilon = 10^{-2}$, $n = 7$ suffices; for $\varepsilon = 10^{-4}$, $n = 14$ suffices.

1.18 (a) Let $f(x) = x - 3^{-x}$. This f is continuous on $\mathbb{R}$, so on any closed interval we might choose. And, $f(0) = -1$ while $f(1) = 2/3$. By the IVT, there is some number c inside the interval $[0, 1]$ for which $f(c) = 0$.

(b) Since $f(x) = 4x^2 - e^x$ is negative at $x = 0$ and positive at $x = 1$, a similar argument involving the IVT shows that $f(x) = 0$ is satisfied at some number x inside $[0, 1]$.

1.21 The basic strategy to approximating $\int_a^b f(x)\,dx$ is this: Find an easily-integrated function $g(x)$ such that

$$|f(x) - g(x)| \;<\; \frac{\varepsilon}{b-a}\,, \qquad \text{for } a \le x \le b\,,$$

where $\varepsilon = 5 \times 10^{-6}$. For, if this is achieved, then

$$\begin{aligned}
|f(x) - g(x)| \;<\; \frac{\varepsilon}{b-a} \qquad &\Rightarrow \qquad -\frac{\varepsilon}{b-a} \;<\; f(x) - g(x) \;<\; \frac{\varepsilon}{b-a} \\
&\Rightarrow \qquad -\int_a^b \frac{\varepsilon}{b-a}\,dx \;<\; \int_a^b [f(x) - g(x)]\,dx \;<\; \int_a^b \frac{\varepsilon}{b-a}\,dx \\
&\Rightarrow \qquad -\varepsilon \;<\; \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \;<\; \varepsilon \\
&\Rightarrow \qquad \left|\int_a^b f(x)\,dx - \int_a^b g(x)\,dx\right| \;<\; \varepsilon
\end{aligned}$$

(a) Since

$$\sin x \;=\; x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \frac{(-1)^{n+1}\cos\xi}{(2n+3)!}x^{2n+3}\,,$$

we have

$$\left|\frac{\sin x}{x} - \left(1 - \frac{1}{3!}x^2 + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n}\right)\right| \;=\; \frac{|\cos\xi|}{(2n+3)!}|x|^{2n+2} \;\le\; \frac{1}{(2n+3)!}\,,$$

for all $x \in [0, 1]$. If we choose $n = 3$, then $1/(2n+3)! \doteq 2.76 \times 10^{-6} < \varepsilon$. Thus, we evaluate

$$\int_0^1 (1 - \frac{1}{6}x^2 + \frac{1}{120}x^4 - \frac{1}{5040}x^6)\,dx \;=\; 1 - \frac{1}{18} + \frac{1}{600} - \frac{1}{35280} \;\doteq\; 0.946083\,.$$

(b) Operating as above, we employ the fact that

$$\cos x \;=\; 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \frac{(-1)^{n+1}\cos\xi}{(2n+2)!}x^{2n+2} \;,$$

to get that

$$\left|\frac{1-\cos x}{x} - \left(\frac{1}{2!}x - \frac{1}{4!}x^3 + \cdots + \frac{(-1)^{n-1}}{(2n)!}x^{2n-1}\right)\right| \;=\; \frac{|\cos\xi|}{(2n+2)!}|x|^{2n+1} \;\le\; \frac{1}{(2n+2)!} \;.$$

This time we must choose $n \ge 4$ in order to get that $1/(2n+2)! < \varepsilon$. Thus, the integral is approximated accurately to 5 decimal places by

$$\int_0^1 \left(\frac{1}{2}x - \frac{1}{24}x^3 + \frac{1}{720}x^5 - \frac{1}{40320}x^7\right) dx \;\doteq\; 0.239812 \;.$$

(c) It can be shown that, for all $x \in [0,1]$,

$$\left|\frac{e^x - 1}{x} - \left(1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots + \frac{1}{n!}x^{n-1}\right)\right| \;=\; \frac{e^\xi}{(n+1)!}|x|^n \;\le\; \frac{e}{(n+1)!} \;<\; \varepsilon \;,$$

when $n \ge 9$. Thus, we take as our approximation to the integral

$$\sum_{j=1}^{9}\int_0^1 \frac{1}{j!}x^{j-1}\,dx \;=\; \sum_{j=1}^{9}\frac{1}{j!\cdot j} \;\doteq\; 1.317902 \;.$$

(d) Working similarly (again with a MacLaurin series for e^x) as in the previous problem, we have

$$\left|\frac{e^x}{x} - \left(\frac{1}{x} + 1 + \frac{1}{2!}x + \frac{1}{3!}x^2 + \cdots + \frac{1}{n!}x^{n-1}\right)\right| \;=\; \frac{e^\xi}{(n+1)!}|x|^n \;\le\; \frac{e^2 2^n}{(n+1)!} \;<\; \varepsilon \;,$$

when $x \in [1,2]$ if $n \ge 12$. Thus, we take as our approximation to the integral

$$\int_1^2 \frac{dx}{x} + \sum_{j=1}^{12}\int_1^2 \frac{1}{j!}x^{j-1}\,dx \;=\; \ln 2 + \sum_{j=1}^{12}\frac{2^j - 1}{j!\cdot j} \;\doteq\; 3.059116 \;.$$