MATH 335: Numerical Analysis

Problem Set 19, Final version

Due Date: Mon., May 4, 2009

Read Sections 11.5–11.9 from Kharab & Guenther, along with your in-class notes.

$\star$39 Consider the IVP of Exercise CP 11.1.2 in Kharab & Guenther:

$$y' = ty^2 , \quad \text{for } t \in [0,2] , \qquad \text{subject to} \qquad y(0) = \frac{2}{5} . \tag{1}$$

A demonstration of the solution of this problem using the *implicit Euler method* was provided in class using this code.

(a) Another possibility is to solve

$$y' = f(t,y) , \quad \text{for } t \in [a,b] , \qquad \text{subject to} \qquad y(a) = \alpha ,$$

using an implicit method based on a trapezoid approximation of the definite integral

$$\int_{t_j}^{t_{j+1}} f(t,y(t))\,dt \approx \frac{t_{j+1}-t_j}{2}[f(t_j,y(t_j)) + f(t_{j+1},y(t_{j+1}))] .$$

That is, given numbers $y_0, y_1, \ldots, y_j$ approximating the solution at mesh points, find y_{j+1} by solving the equation

$$z = y_j + \frac{h}{2}[f(t_j,y_j) + f(t_{j+1},z)] .$$

Write code to find the approximate solution of the IVP (1) employing this method. In my code from class, I solved this equation using Newton's (or the Newton-Raphson) method, taking y_j as my initial estimate of z. Have your code employ Newton's method as well, but take as your initial z-value the estimate that arises from the *explicit Euler method*; that is, at each step begin Newton's method with

$$z_0 = y_j + hf(t_j,y_j) .$$

(b) Modify your code so that, instead of determining y_{j+1} via Newton's method, you do so via a fixed-point iteration. That is, starting with some initial guess z_0, find iterates

$$z_{k+1} = y_j + hf(t_j,z_k) , \qquad k = 0,1,2,\ldots ,$$

until one is satisfied that the difference $|z_{k+1} - z_k|$ is sufficiently small (at which point we take y_{j+1} to be this final computed z_{k+1}).

Note that an initial guess $z_0 = y_j + f(t_j, y_j)$ (as was used in part (a)) represents a *prediction* of the value y_{j+1}, and the process of using it to find a (hopefully improved) value z_1 can be seen as a *correction*. Heun's is a **predictor-corrector method**, in which we stop right then (i.e., we take $y_{j+1} = z_1$), instead of using z_1 as a new prediction and correcting to get z_2, and so on.

(c) Compare the error for the three methods (implicit Euler, the method of part (a), and that of part (b)) at $t = 1$ using these different h-values: $h = 0.2, 0.1, 0.05$. How do these errors compare between methods? What order of convergence do they reveal for each? Is the answer to these questions what you expect?

<u>★40</u> (a) Which of the functions $f(t, y)$ given below satisfy a Lipschitz condition on $D = \{(t, y) \,|\, 0 \le t \le 1,\ y \in \mathbb{R}\}$?

i. $f(t, y) = (1 + t^2)\cos(ty)$

ii. $f(t, y) = g(t)$

iii. $f(t, y) = |y - 1|$

(b) Consider the IVP

$$y' = y^2, \quad t \in [0, 1], \qquad \text{subject to} \qquad y(0) = 2.$$

i. Apply the 4^{th}-order Runge-Kutta method with stepsize $h = 0.01$ to (attempt) to solve this IVP. Plot your "solution."

ii. Explain what is lacking so that our theorem on Existence and Uniqueness of solutions to IVPs does not tell us this solution exists throughout all of $0 \le t \le 1$.

<u>★41</u> (a) Write a routine which, if written in Octave, has declaration

```
function [wtsAB, wtsAM] = adamsWts (n)
```

and, for a given value of $n = 1, 2, 3, 4, 5$, returns in two vectors the appropriate weights (as provided in tables on a class handout) for the n^{th}-order Adams-Bashforth and Adams-Moulton methods.

(b) Implement an Adams-Bashforth-Moulton routine with declaration

```
function [y, t] = abm(f, t0, tlast, y0, order, N)
```

The inputs/outputs here, for the most part, have the same meaning as for the routines in Exercise ★38. The only difference is the `order` input, indicating the desired order of the ABM method. Write your routine so that it employs your RK4 method to solve in the initial steps.