MATH 335: Numerical Analysis

Problem Set 17, Final version

Due Date: Thurs., Apr. 23, 2009

Read Chapter 10 from Kharab & Guenther, along with your in-class notes.

$\underline{\star 35}$ Approximate the integral

$$\int_0^1 \frac{e^x}{\sqrt{x}}\, dx$$

by first making a substitution that turns this improper integral into a proper one.

$\underline{\star 36}$ Write a script in Octave (or other) that employs Gauss-Legendre quadrature to approximate the integral

$$\int_0^1 \int_0^\pi \int_1^2 x^2 \sin(xy\sqrt{z})\, dz\, dy\, dz$$

using $n = 6$ nodes in each "direction".

$\underline{\star 37}$ (a) Let the interval $[a, b]$ be fixed, and suppose

$$D = \left\{(t, x_1, \ldots, x_m) \,|\, a \leq t \leq b,\ -\infty < x_j < \infty \text{ for each } j = 1, 2, \ldots, m\right\}$$

A *sufficient* (though *not necessary*) condition ensuring a function $f(t, x_1, \ldots, x_m)$ is Lipschitz on D is that all of its first partial derivatives are continuous on D and satisfy the inequality

$$\left|\frac{\partial f}{\partial x_j}(t, x_1, \ldots, x_m)\right| \leq L, \qquad \text{for each } j = 1, \ldots, m.$$

Use the *Mean Value Theorem* to show this is true. For simplicity, you may assume (for this part) that $m = 2$.

(b) Our theorem on existence and uniqueness of a solution to the system of IVPs

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(t, \mathbf{x}), \quad t \in [a, b], \qquad \text{subject to} \qquad \mathbf{x}(a) = \boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m),$$

assumes that each of the component functions $f_j(t, x_1, \ldots, x_m)$ of

$$\mathbf{f}(t, \mathbf{x}) = \begin{bmatrix} f_1(t, x_1, \ldots, x_m) \\ f_2(t, x_1, \ldots, x_m) \\ \vdots \\ f_m(t, x_1, \ldots, x_m) \end{bmatrix}$$

is Lipschitz on D. Show that it is equivalent to assume there exists a number M such that $\|\mathbf{f}(t, \mathbf{x}) - \mathbf{f}(t, \mathbf{y})\|_\infty \leq M\|\mathbf{x} - \mathbf{y}\|_1$.

AP 10.6 The **steady state temperature distribution** $u(r,\theta)$ of an insulated disk satisfies the partial differential equation

$$\Delta u \ := \ \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2} \ = \ 0, \qquad 0 \le \theta \le 2\pi, \ \ 0 \le r \le a \ ,$$

$$u(a,\theta) \ = \ f(\theta) \ , \quad f(0) \ = \ f(2\pi) \ , \qquad 0 \le \theta \le 2\pi \ .$$

Here a is the radius of the disk, the origin of the coordinate frame is taken to be at the center of the disk, and $f(\theta)$ is the given temperature on the periphery of the disk. The solution of this problem is

$$u(r,\theta) \ = \ \frac{1}{2\pi}\int_0^{2\pi} \frac{a^2 - r^2}{a^2 - 2ar\cos(\phi - \theta) + r^2}\, f(\phi)\, d\phi \ .$$

For this problem we take $a = 1$.

(a) Let $f(\theta) = 1$. Evaluate the integral at several random points to see that the temperature distribution throughout the disk is probably the constant 1.

(b) Suppose $f(\theta) = \cos^2\theta - \sin^2\theta$. Evaluate the integral at several random points to conclude that $u(r,\theta)$ is probably equal to $r^2(\cos^2\theta - \sin^2\theta)$.

(c) Suppose $f(\theta) = \theta(2\pi - \theta)$, $0 \le \theta \le 2\pi$. Draw the graph of $u(0.2,\theta)$, $u(0.4,\theta)$, $u(0.6,\theta)$, $u(0.8,\theta)$ for $0 \le \theta \le 2\pi$. What is $u(0,\theta)$?

(d) As in part (c), $f(\theta) = \theta(2\pi - \theta)$. Evaluate $u(r,0)$, $u(r,\pi/4)$, $u(r,5\pi/6)$, and $u(r,3\pi/2)$ for $r = 0.9$, 0.95, and 0.99, and observe that they satisfy the estimate

$$0 \ = \ \min_{0\le\theta\le 2\pi} f(\theta) \ \le \ u(r,\theta) \ \le \ \max_{0\le\theta\le 2\pi} f(\theta) \ = \ \pi^2 \ .$$

The estimate is called the **maximum-minimum principle** for solutions to $\Delta u = 0$ in a disk.