

MATH 335: Numerical Analysis

Problem Set 15, Final version

Due Date: Mon., Apr. 13, 2009

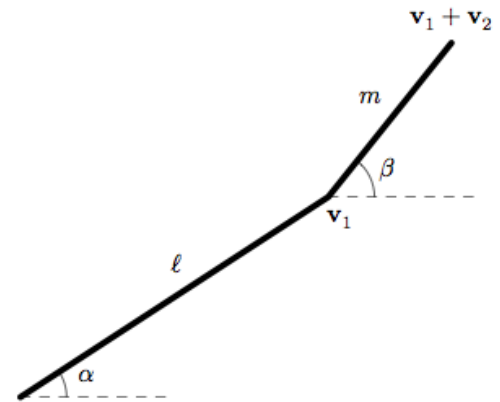
★28 THERMOCLINE DEPTH IN PLATTE LAKE¹. Lakes in the *temperate zone* can become thermally stratified during the summer. Warm, buoyant water near the surface overlies colder, denser bottom water. Such stratification effectively divides the lake vertically into two layers: the *epilimnion* and the *hypolimnion* separated by a plane called the *thermocline*.

Thermal stratification has great significance for environmental engineers studying the pollution of such systems. In particular, the thermocline greatly diminishes mixing between the two layers. As a result, decomposition of organic matter can lead to severe depletion of oxygen in the isolated bottom waters. The location of the thermocline can be defined as the inflection point of the temperature curve—that is, the point at which $d^2T/dx^2 = 0$, while the absolute value of the first derivative or gradient $|dT/dx|$ is a maximum.

The following table gives temperatures measured at various depths in Platte Lake, a lake in the northern part of Michigan's Lower Peninsula near the Sleeping Bear Dunes National Lakeshore. Use cubic splines constructed from this data to approximate the thermocline depth, and the value of the gradient at that depth, for Platte Lake, MI.

| | | | | | | | | |
|------------|------|------|------|------|------|------|------|------|
| Temp. (°C) | 22.8 | 22.8 | 22.8 | 20.6 | 13.9 | 11.7 | 11.1 | 11.1 |
| depth (m) | 0 | 2.3 | 4.9 | 9.1 | 13.7 | 18.3 | 22.9 | 27.2 |

★29 A ROBOT ARM. A (planar) robot arm consists of two rigid rods that are joined end-to-end to a fixed point in the plane, which we take as the origin $\mathbf{0}$. The rods are free to rotate, and the problem is to configure them so that the robot's hand ends up at the prescribed position $\mathbf{a} = (a, b)$. The first rod has length ℓ and makes an angle α with the horizontal, so its end is at position $\mathbf{v}_1 = (\ell \cos \alpha, \ell \sin \alpha)$. The second rod has length m and makes an angle β with the horizontal, and



¹Taken from *Numerical Methods for Engineers, 5th Ed.*, by Steven C. Chapra and Raymond P. Canale, McGraw-Hill, 2006.

so is represented by the vector $\mathbf{v}_2 = (m \cos \beta, m \sin \beta)$. The hand at the end of the second rod is at position $\mathbf{v}_1 + \mathbf{v}_2$, and the problem is to find values for the angles α, β so that $\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{a}$; see the figure above. The problem leads to the system of equations

$$\ell \cos \alpha + m \cos \beta = a, \quad \ell \sin \alpha + m \sin \beta = b,$$

in the unknowns α, β .

- (a) Assume that the rods have lengths $\ell = 2, m = 1$, and that the desired location of the hand is at $\mathbf{a} = (1, 1)$. Find any/all configurations $((\alpha, \beta)$ -pairs and the resulting vectors $\mathbf{v}_1, \mathbf{v}_2$) which make this possible.
- (b) Find a general expression for $\mathbf{J}^{-1}(\alpha, \beta)$, the inverse of the Jacobian matrix, and use it to find an expression for $\mathbf{J}_G(\alpha, \beta)$, the Jacobian of the function \mathbf{G} given by

$$\mathbf{G}(\alpha, \beta) := \begin{bmatrix} \alpha \\ \beta \end{bmatrix} - \mathbf{J}^{-1}(\alpha, \beta) \begin{bmatrix} \ell \cos \alpha + m \cos \beta - a \\ \ell \sin \alpha + m \sin \beta - b \end{bmatrix}.$$

(You may use the values $\ell = 2, m = 1$ as you do this.) Find (approximately) $\rho(\mathbf{J}_G(\alpha, \beta))$ (the spectral radius) at all points you found in part (a). What do you learn about the convergence of Newton's method from these spectral radii?

★30 DE MOIVRE'S FORMULA. Show that, for each real number x and each integer n ,

$$(\cos x + i \sin x)^n = \cos(nx) + i \sin(nx).$$

★31 (a) Write a routine to carry out a *composite Simpson rule*. If written in OCTAVE, the function declaration should be

```
function intApprox = simpson(f, a, b, n)
```

where f is a function handle for the integrand, a, b are the limits of integration in the definite integral $\int_a^b f(x) dx$ for which an approximation is sought, and n is the number of evenly-spaced subintervals to use. You should require $n > 1$, but should *not* require n to be even. Instead, write your routine so that it *checks* if n is even and, if so, employs the *Simpson 1/3 rule* throughout. If, however, n is odd, have the routine employ the *Simpson 3/8 rule* in the final three subintervals and the *1/3 rule* in all others.

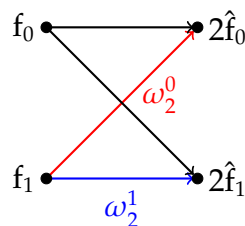
- (b) Test that your routine does, in fact, give reasonable approximations of definite integrals. In particular, check that $\int_{0.1}^{\pi} \frac{\sin x}{x} dx \doteq 1.752$.
- (c) If you try to use your routine in `simpson.m` to approximate the integral $\int_0^{\pi} \frac{\sin x}{x} dx$, what goes wrong?

- ★32 (a) Suppose $\mathbf{f} = (f_0, f_1, f_2, f_3) \in \mathbb{C}^4$. Use information given in class to write out, in long form, the components of $\hat{\mathbf{f}}$, the DFT of \mathbf{f} . To further illustrate what I am after, the expression for the first component is

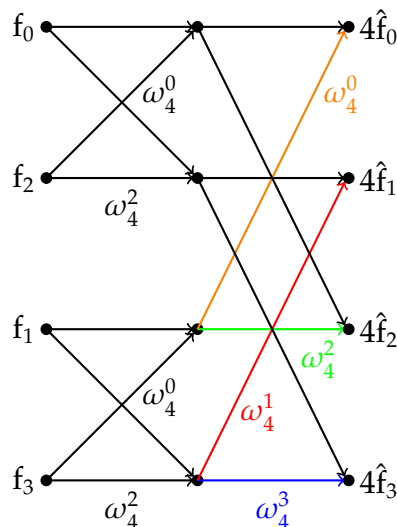
$$\hat{f}_0 = \frac{1}{2} \left[\frac{1}{2}(f_0 + f_2) + \frac{1}{2}(f_1 + f_3) \right] = \frac{1}{4} [(f_0 + f_2) + (f_1 + f_3)] .$$

- (b) Repeat the process of (a), but this time assume $\mathbf{f} = (f_0, f_1)$ (i.e., $\mathbf{f} \in \mathbb{C}^2$).

- (c) The figure at right is meant to depict the result of part (b) diagrammatically. Here ω_2 denotes the primitive square root of unity—i.e., $\omega_2 = -1$; the superscript is an exponent.



This next figure (right) depicts the result of part (a). In this instance, ω_4 denotes the primitive 4th root of unity given by $\omega_4 = e^{-2\pi i/4} = -i$, and so $\omega_4^0 = 1$, $\omega_4^2 = (-i)^2 = -1$, and $\omega_4^3 = (-i)^3 = i$. (If you have printed this document on a grayscale printer, you should view the .pdf file on your computer’s monitor for assistance in determining which arrows are associated with multiplication by powers of ω_4 .)



Draw a similar diagram depicting the DFT for elements in \mathbb{C}^8 .

- (d) Use your diagram from part (c) to calculate, by hand, the 4th component \hat{f}_3 of the DFT of the vector $\mathbf{f} = (1, 2, 3, 4, 5, 6, 7, 8) \in \mathbb{C}^8$. (Here, “by hand” does not rule out the use of OCTAVE as a “calculator” when multiplying complex numbers. When you first power up OCTAVE, the symbol i refers to $\sqrt{-1}$. If you have used i in another context—perhaps as an index in a for loop since powering up, you may have to execute `clear i` in order to restore its original meaning.) To assist you, here are the relevant powers of $\omega_8 = e^{-2\pi i/8}$:

$$\begin{aligned} \omega_8^1 &= \frac{1}{\sqrt{2}}(1 - i) & \omega_8^3 &= -\frac{1}{\sqrt{2}}(1 + i) & \omega_8^5 &= \frac{1}{\sqrt{2}}(-1 + i) & \omega_8^7 &= \frac{1}{\sqrt{2}}(1 + i) \\ \omega_8^2 &= -i & \omega_8^4 &= -1 & \omega_8^6 &= i \end{aligned}$$