MATH 335: Numerical Analysis

Problem Set 13, Final version

Due Date: Thurs., Apr. 2, 2009

Read the handout titled "Best Approximation of Continuous Functions".

$\star$23 (a) In class, we asserted that $g^*(x) = 1.2643 + 1.1752x$ is the best linear approximation of $f(x) = e^x$ on the interval $[-1, 1]$ in the $\|\cdot\|_\infty$-sense. That is, except for the rounding of the coefficients in g^*, it is the case that, given any other linear polynomial $g(x) = mx + b$,

$$\|f - g^*\|_\infty \;\leq\; \|f - g\|_\infty \, .$$

Find the best linear approximation of f on this same interval but in the $\|\cdot\|_2$-sense.

(b) Let g^* continue to be the same as in part (a), and call your answer to part (a) h^*. In class, we said $\|f - g^*\|_\infty \doteq 0.279$. Find the values of

$$\|f - h^*\|_\infty \, , \quad \|f - g^*\|_2 \, , \quad \text{and} \quad \|f - h^*\|_2 \, .$$

Do the resulting numbers fit your expectations of minimality?

$\star$24 In class we considered the problem of finding the best approximating function to f (in the 2-norm sense) in the class Π_k of polynomials of degree k or less. Specifically, we worked out the entries of a matrix equation whose solution yields the coefficients $a_0, a_1, \ldots, a_k$ of the linear combination

$$\sum_{j=0}^{k} a_j \phi_j(x)$$

of basis functions $\{\phi_0, \phi_1, \ldots, \phi_k\}$ for Π_k which is the best approximator on the interval $[a, b]$.

(a) Consider the sequence $(P_n)_{n=0}^{\infty}$ of *Legendre polynomials*. The first 4 of them are

$$P_0(x) = 1 \, , \quad P_1(x) = x \, , \quad P_2(x) = \frac{3x^2 - 1}{2} \, , \quad P_3(x) = \frac{5x^3 - 3x}{2} \, .$$

Show that any $p \in \Pi_3$ may be written as a linear combination

$$p(x) \;=\; b_0 P_0(x) + b_1 P_1(x) + b_2 P_2(x) + b_3 P_3(x) \, .$$

This means that $\{P_0, P_1, P_2, P_3\}$ serves as an alternate basis (to $\{1, x, x^2, x^3\}$) for Π_3.

(b) Take the interval $[a, b]$ to be $[-1, 1]$. Show that $\left\langle P_i, P_j \right\rangle = 0$ for $0 \leq i, j \leq 3$ when $i \neq j$.

(c) Write down the Gram matrix that results from using the Legendre polynomial basis for Π_3. (Evaluate all inner products so that you know the numbers in each entry of this matrix.)

$\star$25 Suppose $f\colon [-1, 1] \to \mathbb{C}$ is continuous and satisfies $f(-1) = f(1)$. For a fixed positive integer k define functions

$$\psi_m(x) \;:=\; e^{im\pi x}\,, \qquad m = -k, -k+1, \ldots, 0, 1, \ldots, k\,,$$

and let $\mathcal{G} = \operatorname{span}(\{\psi_m \,|\, m = -k, \ldots, k\})$. It can be shown that the functions ψ_m, $m = -k, \ldots, k$ are linearly independent, and hence form a basis for $\mathcal{G}$. In class we asserted that an alternate basis for $\mathcal{G}$ is

$$S \;:=\; \{1, \cos(\pi x), \cos(2\pi x), \ldots, \cos(k\pi x), \sin(\pi x), \sin(2\pi x), \ldots, \sin(k\pi x)\}\,,$$

and we showed that if a function $g \in \mathcal{G}$ is already written in terms of the first basis

$$g(x) \;=\; \sum_{m=-k}^{k} c_m \psi_m(x)\,,$$

(i.e., the coefficients c_m are known), then corresponding coefficients a_m, b_m in the expansion

$$g(x) \;=\; \frac{a_0}{2} + \sum_{m=1}^{k} [a_m \cos(m\pi x) + b_m \sin(m\pi x)] \tag{1}$$

(an expansion of g in the alternate basis S) may be determined from

$$b_m \;=\; i(c_m - c_{-m})\,, \quad m = 1, \ldots, k, \qquad \text{and} \qquad a_m \;=\; \begin{cases} 2c_0\,, & \text{if } m = 0\,, \\ c_{-m} + c_m\,, & \text{if } m = 1, \ldots, k\,. \end{cases}$$

(a) Use these expressions for the a_m, b_m, along with

$$c_m \;=\; \frac{1}{2} \int_{-1}^{1} f(x) e^{-im\pi x}\, dx$$

to derive the following expressions for a_m, b_m in terms of the original function f and the basis functions of S:

$$a_m \;=\; \int_{-1}^{1} f(x) \cos(m\pi x)\, dx \;=:\; \langle f, \cos(m\pi\cdot)\rangle\,, \qquad m = 0, 1, \ldots, k,$$
$$b_m \;=\; \int_{-1}^{1} f(x) \sin(m\pi x)\, dx \;=:\; \langle f, \sin(m\pi\cdot)\rangle\,, \qquad m = 1, \ldots, k.$$

(b) In Octave, it is possible to compute numerical approximations of integrals using the `quad` function. Specifically, commands like

```
octave:201> function y = g(x), y = x; end
octave:202> quad(@g, 0, 1)
ans =  0.50000
```

produce the (approximate) value of $\int_0^1 x\,dx$. To calculate integrals like those in the expressions for a_m, b_m in part (a), one might use commands like

```
octave:203> function y = f(x), y = x; end
octave:204> function y = fcos(x, m), y = f(x)*cos(m*pi*x); end
octave:205> function y = fsin(x, m), y = f(x)*sin(m*pi*x); end
octave:206> a2 = quad(@(x) fcos(x, 2), -1, 1)
octave:207> b5 = quad(@(x) fsin(x, 5), -1, 1)
```

Write a routine which presumes you have already defined functions `fcos()`, `fsin()` appropriately (these should rely on a defined function `f()` as above), and whose declaration line looks like

```
function [a, b] = fourierCoeffs(fcos, fsin, k)
```

The inputs include handles to the two functions `fcos()`, `fsin()` and a positive integer k indicating the number of basis vectors to use (an input k results in $(2k + 1)$ basis vectors). (What would be even better is to not require the user to have declared these *three* functions, but only the one (`f()`, and using the declaration

```
function [a, b] = fourierCoeffs(f, k)
```

achieves the same thing as the above.) The outputs of this function should be the Fourier coefficients a_m, $m = 0, 1, \ldots, k$ and b_m, $m = 1, \ldots, k$.

Next write a routine with declaration

```
function y = fsEval(x, a, b)
```

for which the inputs `a`, `b` are vectors of Fourier coefficients such as those returned by `fourierCoeffs.m`. The function should evaluate the function g of equation (1) at the values given in `x`. (It should not be a requirement that the values in `x` all lie inside $[-1, 1]$.)

(c) Consider the function $f(x) = x$. (Note: This function *does not satisfy* the relationship $f(-1) = f(1)$.) Draw the graph of f on the interval $[-1.5, 1.5]$. Then add to this graph plots of the best approximating truncated Fourier series using 5, 11, 19 and 51 basis functions of the form S. As more basis functions are included, what might you conclude about the convergence of the truncated Fourier series to the true f?