MATH 335: Numerical Analysis

Problem Set 11, Final version

Due Date: Mon., Mar. 16, 2009

Read Sections 5.1, 5.3–5.4 and 6.1 in Kharab & Guenther.

★17 Consider the polynomial function $f(x) = x^3 - x^2 + 2x - 7$.

(a) Use the values of f at the four points on its graph corresponding to $x = 1, 3, 4.2$, and 8 to construct a 3rd degree interpolating polynomial. (You may use OCTAVE commands for this as you see fit.) Do you get what you expect?

(b) Now use the ten points from the graph of f corresponding to abscissas $x = 1, 2, \ldots, 10$, and construct a 9th degree interpolating polynomial. What do you observe this is and is not expected? Can you offer any explanations for the unexpected?

5.4.2 Consider the points $(0, 1)$, $(2, -1)$, $(4, 3)$, and $(6, 4)$.

(a) Write the *cardinal functions* associated with these points.

(b) Use these cardinal functions to write the Lagrange form of the interpolating polynomial.

AP 5.4 Consider the function $f(x) = 1/(1 + x^2)$.

(a) Determine the polynomial $p_{10}(x)$ of 10th degree that interpolates the points with abscissa (-5), (-4), . . . , 4, 5, all spaced one unit apart.

(b) Plot both p and f in the interval $[-5, 5]$. Note that the wiggles in p_{10} occur near the ends of the interval over which the interval is used, by finding the error $|f(x) - p_{10}(x)|$ at $x = 4.2$, 4.4, 4.6, and 4.8. This is a characteristic behavior of interpolating polynomials of high degree.

★18 Consider the Vandermonde matrix for n points $x_1, \ldots, x_n$ as given in the text:

$$\mathbf{V} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

(a) Show, in the case where the number of points $n = 2$, that $\det(\mathbf{V}) = x_2 - x_1$.

(b) Show, in the case where the number of points $n = 3$, that

$$\det(\mathbf{V}) = (x_3 - x_1)(x_2 - x_1)(x_3 - x_2) \ .$$

(c) **Optional**. Show, in the case of arbitrary integer $n \geq 2$, $\det(\mathbf{V}) = \prod_{1 \leq j < i \leq n}(x_i - x_j)$.

(d) Use the result of part (c) to show that $\mathbf{V}$ is nonsingular if and only if each of the x_j, $j = 1, \ldots, n$ are distinct.

5.3.7 Imagine a table for $\ln(x)$ with $x \in [0.5, 1]$. How many evenly-spaced entries must we have so that polynomial interpolation (based on these entries) of $\ln x$ in this interval to be accurate to within 10^{-3}?

5.3.8 In this exercise we investigate the error formula of polynomial interpolation. We will try to predict, based on this formula, whether polynomial interpolation provides a good fit for a given function f on $[0, 2]$. For each part,

i. Find a general formula for the n^{th} order derivative of the function. Check whether the n^{th} derivative exists at all points in $[0, 2]$.

ii. Determine a bound on the n^{th} order derivative inside $[0, 2]$, and use it to bound the error $|f(t) - p_n(t)|$, where p_n is the n^{th} degree interpolating polynomial to f on equally-spaced points $0 = x_0 < x_1 < \cdots < x_n = 2$, with each $x_{j+1} - x_j = 2/n$.

iii. Suppose we wish to interpolate f at equally-spaced points in $[0, 2]$; say we start with 5 points, then 10 points, etc. What is your prediction concerning convergence of the resulting p_n to f?

Work with the following functions defined on the interval $[0, 2]$.

(a) $f(x) = \sin x$.

(b) $f(x) = 1/(x + 1)$.

★19 (a) Let $c < d$ be real numbers, and consider the function $f(x) = (x - c)(x - d)$. Use calculus to show that $|f|$ has one maximum inside $[c, d]$, located at $x = (c + d)/2$.

(b) Take $\Psi(x)$ to be the function

$$\Psi(x) \ = \ \prod_{i=1}^{n}(x - x_i) \ ,$$

with $x_{i+1} - x_i = h > 0$ (h fixed) for each $i = 1, \ldots, n - 1$. Show that, for each $t \in [x_1, x_n]$,

$$|\Psi(t)| \ \leq \ \frac{1}{4}(n - 1)! \, h^n \ .$$