MATH 335: Numerical Analysis

Problem Set 10, Final version

Due Date: Thurs., Mar. 12, 2009

Read Section 3.8 in Kharab & Guenther.

3.8.2 Find the Jacobian matrix $\mathbf{J_F}(x, y)$ at the point $(-1, 4)$ for the function $\mathbf{F}\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}(x, y) = \begin{bmatrix} F_1(x, y) \\ F_2(x, y) \end{bmatrix} = \begin{bmatrix} x^3 - y^2 + y \\ xy + x^2 \end{bmatrix}.$$

3.8.5 Solve the nonlinear system of equations

$$\begin{aligned} 10 - x + \sin(x + y) - 1 &= 0 \\ 8y - \cos^2(z - y) - 1 &= 0 \\ 12z + \sin z - 1 &= 0 \end{aligned}$$

using Newton's method with $x_0 = 0.1$, $y_0 = 0.25$ and $z_0 = 0.08$.

3.6.6 Consider the system of two equations

$$\begin{aligned} f_1(x, y) &= x^2 + ay^2 - 1 = 0 \\ f_2(x, y) &= (x - 1)^2 + y^2 - 1 = 0 \end{aligned}$$

where a is a parameter.

(a) Write down two explicit formulas for Newton's iterations for this system. First, write it in the form

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - J^{-1}(x_n, y_n) \begin{bmatrix} f_1(x_n, y_n) \\ f_2(x_n, y_n) \end{bmatrix},$$

where you explicitly exhibit the Jacobian J and its inverse J^{-1}. Second, evaluate this expression explicitly—i.e., multiply it out and simplify.

(b) Using these formulas, write a (very short) OCTAVE/MATLAB program to implement Newton iteration just for this example. Try it for $a = 1$.

★16 Let $S \subset \mathbb{R}^n$ be open and convex (convexity means that for each pair of vectors $\mathbf{x}$, $\mathbf{y} \in S$, the entire line segment $\mathbf{x} + t(\mathbf{y} - \mathbf{x})$, $0 \le t \le 1$ lies in S), and suppose $\mathbf{F}\colon S \to S$, given by

$$\mathbf{F}(\mathbf{x}) \;=\; \mathbf{F}(x_1, \ldots, x_n) \;=\; \begin{bmatrix} F_1(x_1, \ldots, x_n) \\ F_2(x_1, \ldots, x_n) \\ \vdots \\ F_n(x_1, \ldots, x_n) \end{bmatrix} ,$$

is differentiable in S. Let $\mathbf{x}$, $\mathbf{y} \in S$, and define the function $g\colon [0, 1] \to S$ by

$$g(t) \;=\; \mathbf{F}(\mathbf{x} + t(\mathbf{y} - \mathbf{x})) \; .$$

Show that

$$g'(t) \;=\; \mathbf{J}(\mathbf{x} + t(\mathbf{y} - \mathbf{x}))(\mathbf{y} - \mathbf{x}) \; ,$$

where $\mathbf{J}(\mathbf{x})$ is the Jacobian matrix given by

$$\mathbf{J}(\mathbf{x}) \;=\; \mathbf{J}(x_1, \ldots, x_n) \;=\; \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_1}{\partial x_n}(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\mathbf{x}) & \cdots & \frac{\partial F_n}{\partial x_n}(\mathbf{x}) \end{bmatrix} .$$

Note that this is an exercise in using the *chain rule*.