

# MATH 335: Numerical Analysis

## Problem Set 9, Final version

Due Date: Thurs., Mar. 5, 2009

4.3.12 Consider the *sparse* matrix  $\mathbf{A} = (a_{ij})_{1 \leq i, j \leq 10}$  with  $a_{ij} \neq 0$  for  $i = 1, j = 1, \dots, 10$ , as well as  $j = 1, i = 1, \dots, 10$  and  $i = j, j = 1, \dots, 10$ ; otherwise  $a_{ij} = 0$ .

- Sketch the *sparsity pattern* of such a matrix, i.e., where the nonzero entries are.
- Verify that the first step of Gaussian elimination without pivoting changes zero entries into nonzero ones.
- Show that we can avoid transforming zero entries into nonzero ones by simply interchanging rows and/or columns of  $\mathbf{A}$  before applying Gaussian elimination.

★14 In class we described the SOR iterative method for solving  $\mathbf{Ax} = \mathbf{b}$  (where  $\mathbf{A}$  is  $n$ -by- $n$ ) as involving iterations

- Give an initial guess  $\mathbf{x}^{(0)}$ .
- For some fixed  $\omega \in (0, 2)$  and each  $k = 0, 1, 2, \dots$ , solve  $\mathbf{M}_\omega \mathbf{x}^{(k+1)} = \mathbf{N}_\omega \mathbf{x}^{(k)} + \mathbf{b}$  for  $\mathbf{x}^{(k+1)}$ , where  $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F}$ , and

$$\mathbf{M}_\omega := \frac{1}{\omega} \mathbf{D} - \mathbf{E}, \quad \text{and} \quad \mathbf{N}_\omega := \frac{1 - \omega}{\omega} \mathbf{D} + \mathbf{F},$$

- Write a routine which accepts a square matrix  $\mathbf{A} = \mathbf{D} - \mathbf{E} - \mathbf{F}$ , and returns a near-best  $\omega$ -value as determined by the spectral radius of  $\mathbf{B}_\omega = \mathbf{M}_\omega^{-1} \mathbf{N}_\omega$ . As you write this routine, you may wish to employ one or more of the following OCTAVE commands: `max()`, `eig()`, `abs()`, `triu(A, 1)`, `tril(A, -1)`, `diag(diag(A))`.
- Write a routine called `sor.m` that implements the SOR method for solving  $\mathbf{Ax} = \mathbf{b}$ . Your routine should accept as input the matrix  $\mathbf{A}$ , the right-hand side vector  $\mathbf{b}$ , an initial approximation  $\mathbf{x}^{(0)}$  of the solution, a user-specified tolerance `tol`, and a user-specified maximum number  $m$  of iterations. Your routine should first call the routine you wrote for part (a) to determine if there is some value of  $\omega$  for which the iteration should converge, refusing to go on (and displaying some sort of message) if there is none. Assuming some reasonable  $\omega$ -value is found then, subject to the constraint that it does not iterate more than  $m$  times, the routine should go until one of these two criteria is met:

$$\|\mathbf{Ax}^{(k)} - \mathbf{b}\|_2 < \text{tol}, \quad \text{or} \quad \|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_2 < \text{tol}.$$

★15 Consider the matrix system

$$\begin{bmatrix} 2 & -5 & 2 & 5 & 30 & -8 \\ -8 & 36 & -1 & 0 & 1 & 8 \\ 11 & -4 & 25 & 1 & -4 & 4 \\ 1 & 0 & -3 & 19 & 2 & 1 \\ 42 & -2 & -9 & 0 & 3 & 0 \\ -3 & 7 & -7 & 4 & 5 & -32 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 27 \\ 36 \\ 42 \\ 18 \\ 54 \\ 60 \end{bmatrix}.$$

- (a) In its current form, the coefficient matrix does not appear to be diagonally dominant. Find a related matrix problem which is.
- (b) Solve your related problem using each of three iterative methods: Jacobi, Gauss-Seidel, and SOR (employing your routine of the previous problem). Compare the number of steps until an acceptable solution is reached.

AP 4.6 Consider heat conduction in a small wire carrying electrical current that is producing heat at a constant rate. The equation describing the temperature  $y(x)$  along the wire ( $0 \leq x \leq 1$  cm) is

$$D \frac{d^2 y}{dx^2} = -S,$$

with boundary conditions  $y(0) = y(1) = 0^\circ\text{C}$ , thermodiffusion coefficient  $D = 0.01$   $\text{cm}^2/\text{sec}$ , and normalized source term  $S = 1^\circ\text{C}/\text{sec}$ .

If we discretize the domain into 20 equal sub-intervals, using  $x_j = j/20$  for  $j = 0, 1, \dots, 20$ , we can approximate the equation at  $x_j$  to obtain

$$D \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} = -S,$$

where  $y_j$  is the temperature at  $x = x_j$  and  $h = 0.05$  is the step size. If we apply the boundary conditions at  $x_0$  and at  $x_{20}$  we are left with 19 equations for 19 unknown temperatures,  $y_1, \dots, y_{19}$ .

(a) Use the particulars of this problem to derive the matrix equation  $\mathbf{A}\mathbf{y} = \mathbf{b}$ , where

$$\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{18} \\ y_{19} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -0.25 \\ -0.25 \\ -0.25 \\ \vdots \\ -0.25 \\ -0.25 \end{bmatrix}.$$

(b) Solve the above steady-state system using an iterative method and starting value  $\mathbf{y} = \mathbf{0}$ .