MATH 335: Numerical Analysis

Problem Set 8, Final version

Due Date: Mon., Mar. 2, 2009

<u>4.3.6</u> Find the scale vector (labeled **c** in the text) and the third pivot row if Gaussian elimination with scaled partial pivoting is used on the matrix

$$\begin{bmatrix} -2 & 4 & 5 & -1 \\ -1 & 2 & 9 & 4 \\ 6 & 2 & -9 & -5 \\ 3 & 0 & 3 & -8 \end{bmatrix}.$$

<u>4.3.10</u> The Hilbert matrix **A** of order n (see the OCTAVE/MATLAB command `hilb(n)`) is defined by

$$a_{ij} = \frac{1}{i+j-1}, \qquad i, j = 1, 2, \ldots, n.$$

It is the classical example of an ill-conditioned matrix: small changes in its entries will produce a large change in the solution to the system $\mathbf{Ax} = \mathbf{b}$.

(a) Solve the system $\mathbf{Ax} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ for $n = 3$ using Gaussian elimination. (Try it both using our routine from Exercise ★9(c), and using the backslash command in OCTAVE.)

(b) Approximate the Hilbert matrix to three decimal digits— that is, take

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1.0 & 0.5 & 0.333 \\ 0.5 & 0.333 & 0.25 \\ 0.333 & 0.25 & 0.2 \end{bmatrix},$$

—and solve the system $\tilde{\mathbf{A}}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Compare with the solution obtained in part (a).

<u>CP 4.3.8</u> To decompose the expression $(4x^3 + 4x^2 + x - 1)/[x^2(x+1)^2]$ into a sum of partial fractions, we write

$$\frac{4x^3 + 4x^2 + x - 1}{x^2(x+1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1} + \frac{A_4}{(x+1)^2}.$$

The coefficients A_i, for $i = 1, \ldots, 4$, are given by the solution of a system of equations. Write the system of equations in a matrix form and then use Gaussian elimination to find these coefficients.

$\star$11 Use Octave's `lu()` command to obtain a factoriation **LU** = **PA**, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{bmatrix}.$$

(You may wish to enter the command "`format rat`" first.) Then use it to find $\mathbf{A}^{-1}$.

$\star$12 We have discussed various norms for vectors in $\mathbb{R}^n$. The three most popular are $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$ (the 1-, 2- and ∞-norms).

(a) Make a careful sketch of the set $\left\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_p = 1\right\}$ for the three cases $p = 1$, 2 and ∞. (Imagine these collections of vectors in *standard position* (i.e., with their tails placed at the origin), and draw in only where their heads would be (i.e., the terminal points).

(b) Estimate (perhaps using your sketches) positive numbers c, C, γ, Γ, λ, Λ such that

$$c\|\mathbf{v}\|_1 \le \|\mathbf{v}\|_2 \le C\|\mathbf{v}\|_1\,, \qquad \gamma\|\mathbf{v}\|_2 \le \|\mathbf{v}\|_\infty \le \Gamma\|\mathbf{v}\|_2\,,$$

and

$$\lambda\|\mathbf{v}\|_\infty \le \|\mathbf{v}\|_1 \le \Lambda\|\mathbf{v}\|_\infty\,, \qquad \text{for all } \mathbf{v} \in \mathbb{R}^2\,.$$

$\star$13 Let $\|\cdot\|$ be some (fixed) norm on $\mathbb{R}^n$, and for any n-by-n matrix **A** let $\|\mathbf{A}\|$ denote the subordinate matrix norm

$$\|\mathbf{A}\| := \max\{\|\mathbf{A}\mathbf{v}\| : \|\mathbf{v}\| = 1\}\,.$$

(a) **Optional**. Show that, indeed, the following inequalities hold:

$$\|\mathbf{A}\mathbf{w}\| \le \|\mathbf{A}\|\,\|\mathbf{w}\| \qquad \text{for all } \mathbf{w} \in \mathbb{R}^n \text{ and } n\text{-by-}n \text{ matrices } \mathbf{A},$$
$$\|\mathbf{A}\mathbf{B}\| \le \|\mathbf{A}\|\,\|\mathbf{B}\| \qquad \text{for all } n\text{-by-}n \text{ matrices } \mathbf{A} \text{ and } \mathbf{B}\,.$$

(b) Show that, using *any* natural matrix norm, $\|\mathbf{I}\| \ge 1$, where **I** is the n-by-n identity matrix. (Hint: Compare the norms of **I** and $\mathbf{I}^2$.) Then show that cond(**A**) ≥ 1 for any n-by-n matrix **A**.

(c) Suppose we have a sequence $(\mathbf{B}_k)$ of n-by-n matrices. The statement "$\mathbf{B}_k \to \mathbf{B}$" is taken to mean that $\lim_{k\to\infty} \|\mathbf{B}_k - \mathbf{B}\| = 0$ under some (any) matrix norm $\|\cdot\|$. Show that, for a given matrix **A**, if there is an *subordinate* matrix norm $\|\cdot\|$ under which **A** satisfies $\|\mathbf{A}\| < 1$, then $\mathbf{A}^k \to \mathbf{0}$ (i.e., the sequence $\mathbf{A}$, $\mathbf{A}^2$, $\mathbf{A}^3$, ... of powers of **A** converges to the zero matrix).

(d) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1/2 & 2 \\ 0 & 1/2 \end{bmatrix}.$$

Use OCTAVE's `norm` command to find the value of $\|\mathbf{A}\|_p$ for $p = 1, 2$ and ∞. In light of the result from the previous part, you might *suspect* that $\mathbf{A}^k \nrightarrow \mathbf{0}$. Show, however, that the sequence $(\mathbf{A}^k)$ does, indeed, converge to $\mathbf{0}$.