

MATH 335: Numerical Analysis

Problem Set 4, Final version

Due Date: Mon., Feb. 16, 2009

Read Sections 9.2 and 3.1 in Kharab & Guenther.

★3 Richardson extrapolation to get higher-order approximations. Let f be a function having derivatives of all orders. For given numbers x and h , we have defined $a(\cdot)$ to be the centered difference formula

$$a(h) := \frac{f(x+h) - f(x-h)}{2h} .$$

In class we used Taylor expansions of $f(x+h)$ and $f(x-h)$ to show that

$$f'(x) = a(h) + c_2h^2 + c_4h^4 + c_6h^6 + \dots . \quad (1)$$

Here, the coefficients c_j depend on f and x , as does the function $a(h)$. Still working with the same f and x , we have

$$f'(x) = a(h/2) + c_2(h/2)^2 + c_4(h/2)^4 + c_6(h/2)^6 + \dots , \quad (2)$$

$$f'(x) = a(h/4) + c_2(h/4)^2 + c_4(h/4)^4 + c_6(h/4)^6 + \dots . \quad (3)$$

(a) In class we used equations (1) and (2) to arrive at the new approximation

$$\begin{aligned} f'(x) &= \frac{4}{3}a(h/2) - \frac{1}{3}a(h) - \frac{1}{4}c_4h^4 - \frac{5}{16}c_6h^6 - \dots \\ &= D_{2,2} + d_4h^4 + d_6h^6 + \dots . \end{aligned} \quad (4)$$

Use equations (2) and (3) to derive the similar equation

$$f'(x) = D_{3,2} + \frac{1}{16}d_4h^4 + \frac{1}{64}d_6h^6 + \dots . \quad (5)$$

(b) Use equations (4) and (5) to derive another approximation for $f'(x)$ involving $D_{2,2}$ and $D_{3,2}$, whose truncation error is $O(h^6)$. Is this approximation $D_{3,3}$?

9.2.10 The following data gives approximations to the integral $I = \int_a^b f(x) dx$ for a scheme with error terms $E = K_1h + K_2h^3 + K_3h^5 + \dots$:

$$I(h) = 2.3965 , \quad I(h/3) = 2.9263 , \quad I(h/9) = 2.9795 .$$

Construct an extrapolation table to obtain a better approximation.

CP 9.2.2 As provided on the CD accompanying the text, `derive.m` prints a table similar to Table 9.2 on p. 312, but returns no values. Modify the routine so that the values in the 3rd column and beyond are returned in a matrix. Then use this program to approximate the derivative of the following functions:

(a) $f(x) = 3x^3 - 5x^2$, with $h = 0.2$, $N = 4$, at $x = 2.6$.

(b) $f(x) = \sqrt{x^2 + 1}$, with $h = 0.5$, $N = 4$, at $x = 1.3$.

(c) $f(x) = x \cos x - 3x$, with $h = 0.2$, $N = 4$, at $x = 1.4$.

(d) $f(x) = \sin x + x$, with $h = 0.3$, $N = 4$, at $x = \pi/4$.

(e) $f(x) = xe^x - x^2$, with $h = 0.2$, $N = 4$, at $x = 2.3$.

AP 9.3 The bottom graph provided on p. 317 of our text shows the amount of water in a city water tank during one day when no water was pumped into the tank. Approximate the rate of change of water usage at $t = 8, 12$ and 16 .

3.1.8 The function

$$f(x) = x^4 - 8.6x^3 - 35.51x^2 + 464.4x - 998.46$$

has a simple root in the interval $[6, 8]$ and a double root in the interval $[4, 5]$. Use the bisection method to find both roots. (Hint: When a function has a double root, what will be true about its derivative there?)

3.1.13 Consider the function

$$f(x) = (x - 2)^2 - \ln x$$

on the interval $[1, 2]$. (In OCTAVE, the `log()` is really the natural log.)

- Prove that there is exactly one zero of this function in this interval.
- Use the bisection method to approximate a root to 6 (decimal) digit accuracy. (You are welcome to use the `bisect.m` program included on the book's CD. However, force yourself to understand the details well enough to carry out several steps by hand if need be.)
- How many iterates of the bisection method are needed to find an approximation to the root of $f(x) = 0$ in the interval to within an accuracy of 10^{-4} ?