MATH 333: Partial Differential Equations
Problem Set 11, Final version
Due Date: Fri., Dec. 10, 2010

Read Olver, Section 4.4 (material supplemented by Lessons 23 and 41 in Farlow’s book), and Section 10.5.

Most of the problems that follow are required as homework. At the end, however, you have the following two options: Do Problems ⋆35 and 10.2.7 (they come as a pair, continuing to focus on finite difference schemes), or Problem ⋆36 (a FEM problem).

⋆34 In class we started with an operator equation

\[ K[u] = f, \quad 0 < x < \ell, \quad u \in D := \{ v \in C^2([0, \ell]) | v(0) = 0 = v(\ell) \}, \]

where the operator \( K \) is symmetric and positive definite, and “walked” through the Rayleigh-Ritz approach of formulating a finite element (matrix) problem to approximate its solution. Specifically, we saw that solving the problem \( K[u] = f \) was equivalent to minimizing a quadratic functional \( Q(u) \). After assuming the special form \( u = \sum c_i \phi_i \) (which, in most cases, assures we will not find the true solution \( u \)), we found the appropriate coefficients \((c_1, \cdots, c_n)\) for our approximate solution by solving the matrix problem \( Mc = b \), where \( m_{ij} = \langle \phi_i, K[\phi_j] \rangle \) and \( b_i = \langle f, \phi_i \rangle \).

In the study of finite element methods one encounters a slightly different formulation (though the resulting finite-dimensional problem is very often the same), one associated with the name Galerkin. In this problem we investigate Galerkin’s formulation. For this problem, assume the operator \( K \) in (1) has the special form

\[ K[u] := -\frac{d}{dx} \left( k(x) \frac{du}{dx} \right), \quad \text{where } k(x) > 0, \quad \text{for } x \in [0, \ell]. \]

(a) Let \( \langle \cdot, \cdot \rangle \) denote the inner product of \( L^2(0, \ell) \), and define

\[ a(u, v) := \int_0^\ell k(x) u'(x)v'(x) \, dx. \]  

(2)

Show that, if \( u \) satisfies (1), then

\[ a(u, v) = \langle f, v \rangle, \]  

for every \( v \in D \). Notice that (3) makes sense even when we weaken the requirements on \( u, v \) (heretofore it has been that \( u, v \in D \)), demanding, for instance, only that they belong to the set

\[ A := \{ v: [0, \ell] \to \mathbb{R} | v \text{ is continuous}, v' \text{ is PWC and bdd., and } v(0) = 0 = v(\ell) \}. \]
Because of this observation, equation (3), coupled with the requirement that $v \in \mathcal{A}$, is called the weak form of the BVP (1). (Some call it the principle of virtual work.) Any function $u \in \mathcal{A}$ that satisfies (3) no matter the choice $v \in \mathcal{A}$ is called a weak solution of (1).

(b) Why does one bother adding the qualifier weak in the name “weak solution”? Why not simply call it a solution of (1)?

(c) In part (a), you showed that if $u$ is a strong solution of (1) (i.e., it comes from $\mathcal{D}$ and satisfies the differential equation at each $x \in (0, \ell)$), then it satisfies (3) and, hence, is a weak solution. For this part show that if $u$ is a weak solution (i.e., comes from $\mathcal{A}$ and satisfies (3) for every $v \in \mathcal{A}$) and, in fact, satisfies the stronger smoothness condition that it lie in $\mathcal{D}$, then $u$ is a strong solution of (1).

(d) Show that the function $a(\cdot, \cdot)$ defined in (2) is an inner product on $\mathcal{A}$.

(e) As in the Rayleigh-Ritz approach, Galerkin now restricts our attention to a finite-dimensional subspace $\mathcal{S}$ of $\mathcal{A}$. Specifically, let $\{\phi_1, \ldots, \phi_n\}$ be a linearly independent collection of functions selected from $\mathcal{A}$, and let $\mathcal{S}$ be the subspace spanned by these $\phi_i$’s—that is $\mathcal{S}$ consists of precisely those functions which are linear combinations of the $\phi_i$’s. Convince yourself that, if $u$ is a weak solution of (1), then

$$a(u, v) = \langle f, v \rangle, \quad \text{for every } v \in \mathcal{S}. \quad (4)$$

Now, if we find a finite element (approximate) solution $\widetilde{u} \in \mathcal{S}$ (i.e., one that is a linear combination of the $\phi_i$’s) that satisfies

$$a(\widetilde{u}, v) = \langle f, v \rangle, \quad \text{for every } v \in \mathcal{S}, \quad (5)$$

it is going to be the case (in all but the most exceptional circumstances) that $\widetilde{u}$ is different than the weak solution $u$. However, from (4) and (5) it follows that

$$a(u - \widetilde{u}, v) = 0, \quad \text{for all } v \in \mathcal{S}.$$

Considering that $a(\cdot, \cdot)$ is an inner product on $\mathcal{A}$ (and hence on $\mathcal{S} \subset \mathcal{A}$), what does this say, geometrically speaking, about $u$ and $\widetilde{u}$?

(f) The finite element solution is in $\mathcal{S}$, which means it has the form $\widetilde{u} = \sum_i \tilde{c}_i \phi_i$. Since $a(\cdot, \cdot)$ has the linearity properties of an inner product,

$$a(\widetilde{u}, \phi_j) = a\left(\sum_i \tilde{c}_i \phi_i, \phi_j\right) = \sum_{i=1}^n \tilde{c}_i a(\phi_i, \phi_j).$$

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Thus, (5) gives us that
\[ \sum_{i=1}^{n} \tilde{c}_i a(\phi_i, \phi_i) = \langle f, \phi_i \rangle, \quad \text{for } i = 1, 2, \ldots, n. \]

This is \( n \) equations in the \( n \) unknowns \( \tilde{c}_1, \ldots, \tilde{c}_n \). Write this system of equations in matrix form \( A\tilde{c} = d \)—that is, determine the entries of the matrix \( A \) and the vector \( d \).

(g) In class, we found the stiffness matrix \( M \) and load vector \( b \) that arose from the Rayleigh-Ritz approach in the special instance where \( K[u] = -d^2 u/dx^2 \). Show that, in this instance, the matrices \( A \) and \( M \) from the two approaches are the same, as well as the load vectors \( d \) and \( b \).

**4.4.3** Consider the partial differential equation
\[ yu_{xx} + (x + y)u_{yy} = 0. \]

At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

**4.4.5** Steady flow past an airplane is modeled by the partial differential equation
\[ (m^2 - 1)u_{xx} + u_{yy} = 0, \]
in which \( x \) is the flight direction, \( y \) the transverse direction, and \( m \geq 0 \) is the Mach number—the ratio of the airplane’s speed to the speed of sound. Show that the equation is hyperbolic for subsonic flight, but elliptic for supersonic flight.

**10.4.1** Suppose you wish to numerically solve the IBVP
\[ u_{tt} = 8u_{xx}, \quad 0 < x < 3, \quad t > 0, \]
subject to \( BCs: \ u(t, 0) = 0 = u(t, 3), \)
\[ u_t(0, x) = \begin{cases} 1 - 2|x - 1|, & \frac{1}{2} \leq x \leq \frac{3}{2}, \\ 0, & \text{otherwise}, \end{cases} \]
\[ ICs: \ u(0, x) = 0, \]
using the finite difference algorithm given in class (which appears as equations (10.61) and (10.57)—or, equivalently, (10.60) and (10.54)—of our text) with spatial step size \( \Delta x = 0.1 \). What range of time steps \( \Delta t \) are allowed? Test your answer by implementing the numerical solution for one value of \( \Delta t \) in the allowable range and
one value outside. Discuss what you observe in your numerical solutions. In the stable range, compare your numerical solution with that obtained with the smaller (spatial) step size $\Delta x = 0.005$.

Also, investigate Exercise 10.4.2 (no work for it need be submitted) to the point of seeing whether any real differences arise in its solution.

★35 Consider the ordinary differential equation/BVP (a 1-dimensional version of the Poisson problem)

$$- y''(x) = f(x), \quad \text{subject to BCs } y(0) = 0, \ y'(1) = 1. \quad (6)$$

Let us subdivide the interval $[0, 1]$ into $n$ equally-spaced subintervals,

$$0 = x_0 < x_1 < \ldots < x_n = 1, \quad \text{with each } \ x_m - x_{m-1} = h := \frac{1}{n}. \quad (6)$$

(a) If we use a centered difference approximation for $y''(x)$ then, writing $y_m$ as an approximation to the value of $y(x_m)$, derive the finite difference equation

$$- y_{m-1} + 2y_m - y_{m+1} = h^2 f(x_m), \quad (7)$$

holding for $m = 1, 2, \ldots, n - 1$.

(b) While equation (7) provides most of the particulars for a finite difference scheme to approximate the solution of Problem (6), we will not have a complete scheme until we specify how to handle the Neumann boundary condition at $x = 1$. Consider two different schemes, based on the following two ways of dealing with this Neumann condition. In the simpler of the two schemes $S_1$, we approximate $y'(1)$ using a backward difference:

$$\frac{y_n - y_{n-1}}{h} = 1.$$ 

In scheme $S_2$, we

- imagine there to be a ghost node $x_{n+1} = 1 + h$, outside the right end point (where the Neumann condition occurs) of our interval $[0, 1]$,
- require equation (7) to hold at the right end point $x_n$ (i.e., to hold with $m = n$, which gives rise to an equation involving the ghost node $x_{n+1}$), and
- introduce the constraint (approximate equation for $y'(1) = 1$)

$$\frac{y_{n+1} - y_{n-1}}{2h} = 1.$$
Schemes $S_1$ and $S_2$ both may be turned into matrix problems $A_1v_1 = b_1$ and $A_2v_2 = b_2$, where $v_1 = (y_1, y_2, \ldots, y_n)$ and $v_2 = (y_1, y_2, \ldots, y_n, y_{n+1})$ represent the vector of unknowns. Determine the entries of the matrices $A_1, A_2$ and the right-hand sides $b_1, b_2$.

[c] **This problem part is optional.** Are the matrices $A_1, A_2$ symmetric? positive definite? to these concepts. (The definition of “the matrix $A$ is positive definite” is that $v^TAv \geq 0$ for all vectors $v$ of appropriate size, with equality only when $v = 0$.) If the answers are not obvious, then you may resort to use of numerical evidence. Be clear about what you know to be true (from numerical evidence), and what you only guess at being true. Note: Positive definite matrices are always nonsingular, making a problem like $Av = b$ uniquely solvable.

(d) Let $f(x) = -e^{x-1}$, so that the resulting true solution of (6) is $y(x) = e^{-1}(e^x - 1)$. Write algorithms to compute finite difference solutions (with this $f$) both for scheme $S_1$ and $S_2$. Compute the maximum error between true and approximate solutions (i.e., maximum difference $|y(x_m) - y_m|$ taken over all grid points $x = x_m$) using both schemes at several different choices of $h$ (say, $h = 0.1$ and $h = 0.05$, at the least). What can you say about the order of convergence for each scheme? (Recall that, some time ago, in the context of understanding the concept of order of convergence for various finite difference approximations of the first derivative—forward, backward, and centered—we looked at a spreadsheet like this Excel sheet; the basic idea is to see what effect halving the size of $h$ has on the error.)

10.2.7 How would you modify our finite difference schemes for PDEs in order to deal with Neumann boundary conditions? In particular, focus on the implicit scheme for solving the heat problem

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

subject to

$$\begin{cases} 
BCs: & u_t(t,0) = 0 = u_t(t,1), \\
IC: & u(0,x) = x(x - 1/4)(x - 2/3).
\end{cases}$$

Derive a matrix problem $Au^{(j+1)} = b^{(j+1)}$ to solve, indicating the entries in both $A$ and $b^{(j+1)}$, and implement a routine that solves it. If

$$H(t) := \int_0^1 u(t,x) \, dx$$

is a measure of the total heat at time $t$, then for homogeneous Neumann conditions this total heat should not vary with time (i.e., $H(t)$ should be constant). How closely does your numerical solution maintain this constancy of total heat?
Consider the ODE/BVP
\[ y'' + y = -1, \quad 0 < x < 1, \] subject to homogeneous Dirichlet BCs \( y(0) = 0 = y(1) \). \( \text{(8)} \)

(a) Show that, under the inner product of \( L^2(0,1) \), the differential operator \( K = (1 + d^2/dx^2) \) with these BCs is self-adjoint.

(b) Though \( K \) is not positive definite, write out the associated functional
\[ Q[v] := \frac{1}{2} \langle v, K[v] \rangle - \langle f, v \rangle \]

and “simplify” it down to an expression requiring the least amount of differentiability for the function \( v \).

(c) Though \( K \) is not positive definite, apply the finite element method to solve it anyway, looking for approximate solution \( \hat{y} \in \text{span}(\{\phi_1, \ldots, \phi_n\}) \), where the \( \phi_j \), \( j = 1, \ldots, n \), are the hat functions introduced in class. More specifically, let these hat functions be continuous and piecewise linear on the subintervals arising from the uniform partition
\[ 0 = x_0 < x_1 < x_2 < \cdots < x_{n+1} = 1 \]

(that is, each \( x_k - x_{k-1} = h \) for some fixed \( h > 0 \)). Determine the stiffness matrix \( M \) and load vector \( b \), and for \( n = 19 \), solve the matrix problem (perhaps using software to assist you), plotting the resulting approximate solution in \([0,1]\).

(d) The true solution of Problem (1) is
\[ y(x) = -1 + \cos x + \frac{1 - \cos(1)}{\sin(1)} \sin x. \]

Use this information to find the maximum absolute error in our solution from part (c) taken over all grid points.

(e) Repeat parts (c) and (d) for two more mesh sizes, and determine the order of convergence if, indeed, there seems one to be found here.