MATH 333: Partial Differential Equations
Take-Home Test
Due Date: Tues., Nov. 23, 2010

For these problems, you are to work alone, consulting no fellow student nor authority except (possibly) class notes, homework exercises, our texts, standard software that has come up elsewhere in the course (like MATLAB or Mathematica), materials provided by Professor Scofield (Octave code, solutions to homework exercises, etc.), and Professor Scofield himself.

⋆31 Find a series solution of the heat problem

\[ u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \text{subject to} \]

\[ \begin{aligned}
    & \text{BCs: } u(t,0) = 0, \\
    & u(t,1) = \sin t, \\
    & \text{IC: } u(0,x) = 0.
\end{aligned} \]

⋆32 The IBVP for the damped wave equation,

\[ u_{tt} + ku_t = c^2 u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \text{subject to} \]

\[ \begin{aligned}
    & \text{BCs: } u(t,0) = 0, \\
    & u(t,1) = 0, \\
    & \text{ICs: } u(0,x) = f(x), \\
    & u_t(0,x) = 0,
\end{aligned} \]

governs the displacement of a string immersed in a fluid. The string has unit length and is tied at its ends; its initial displacement is \( f \), and it has no initial velocity. The constant \( k \) is the damping constant, assumed to be positive. Use separation of variables to find the solution in the case \( k < 2\pi c \).

⋆33 Consider the Poisson problem with Dirichlet BCs in a square region of the plane

\[ \Delta u = f(x,y), \quad 0 < x < 1, \quad 0 < y < 1, \quad \text{subject to BCs} \]

\[ \begin{aligned}
    & u(x,0) = \sin(\pi x), \\
    & u(x,1) = e^{-2} \sin(\pi x), \\
    & u(0,y) = 0 = u(1,y).
\end{aligned} \]

where \( f(x,y) = (4 - \pi^2)e^{-2y} \sin(\pi x) \).

(a) Employing a finite difference scheme, solve this problem on a grid that uses \( N = 4 \) interior values on the \( x \)-axis and \( M = 4 \) interior values on the \( y \)-axis
to obtain the steady-state temperature distribution. Graph the resulting approximate solution as a surface in 3D. I encourage you to rely heavily on my class examples which used scripts like psset1.m, psset2.m, psset3.m, psset4.m, poissonSolver.m and discreteLaplacian.m, available to download from

http://www.calvin.edu/~scofield/courses/m333/materials/octave/

Pay special attention to how a forcing function is implemented in psset4.m (i.e., the form of its inputs and outputs). Hand in a printout of your code and the graphs you produce, and send me electronic versions as well.

(b) Demonstrate that the true solution of the problem is $u(x, y) = e^{-2y} \sin(\pi x)$, and explain why the problem is, therefore, well-posed.

(c) Compare the values of the true solution with those from your approximate solution in part (a). Among all interior grid points find both the maximum error and the maximum relative error of the numerical solution.

(d) Redo parts (a) and (c), this time doubling $N$ and $M$ so that they are both 8.