Functions of the form \( x^r \), where \( r \) is a (fixed) real number are called “power functions”. When \( r \geq 0 \), the resulting power function is continuous on all of \([0, 1]\), and using some of the theorems of Calculus, it is easy to establish that the function is, therefore, a member of \( L^2(0, 1) \). But what of the power functions with \( r < 0 \)? They are continuous on the open interval \((0, 1)\), but not on the closed interval \([0, 1]\). Are any of them members of \( L^2(0, 1) \)? For which choices of \( r < 0 \)? Note that, for \( r < 0 \),
\[
\int_0^1 x^{2r} \, dx
\]
is an improper integral.

(a) Below are three pairs of vectors in \( \mathbb{R}^2 \), given using Octave notation.
\[
\begin{align*}
\mathbf{u}_1 &= [3; 4], \quad \mathbf{u}_2 = [-4; 3] \\
\mathbf{u}_1 &= [1; 0], \quad \mathbf{u}_2 = [0; -2] \\
\mathbf{u}_1 &= [1; 0], \quad \mathbf{u}_2 = [1; 1]
\end{align*}
\]
For each different pair of vectors, do the following

(i) Determine the lengths (norms) of \( \mathbf{u}_1, \mathbf{u}_2 \).

(ii) Write, if possible, the vector \( \mathbf{v} = [-8; 5] \) as a linear combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \)—that is, find scalars \( c_1, c_2 \) such that
\[
\mathbf{v} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 .
\]

(iii) Determine the inner products of \( \mathbf{v} \) (given above) with \( \mathbf{u}_1, \mathbf{u}_2 \).

(iv) Indicate whether your answers to parts (i) and (iii) might aid in answering part (ii). Why is the answer sometimes yes and sometimes no?

(b) In Calculus we learn about the vector projection of one vector \( \mathbf{v} \) onto another vector \( \mathbf{u} \), given by
\[
\text{proj}_u \mathbf{v} := \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\| \mathbf{u} \|^2} \mathbf{u} .
\]
How is what you were doing in part (a) related to this concept?
(c) [Note: Hold off doing this part until after we discuss the space $L^2(a, b)$ in class.]

Projections are fundamental in the function space $L^2(a, b)$ as well:

$$\text{proj}_g f := \frac{\langle g, f \rangle}{\|f\|^2} f.$$ 

Suppose we are working in $L^2(0, 1)$ and $f(\cdot) = \sin(n\pi \cdot)$ for some integer $n \geq 1$. Show that, no matter the choice of $n$,

$$\frac{\langle g, f \rangle}{\|f\|^2} = 2 \int_0^1 g(x) \sin(n\pi x) \, dx.$$ 

3.1.2(b) Find all separable eigensolutions (i.e., solutions of the form $u(t, x) = e^{\lambda t} v(x)$) to the heat equation $u_t = u_{xx}$ on the interval $0 \leq x \leq \pi$ subject to mixed boundary conditions $u(t, 0) = 0$, $u_x(t, \pi) = 0$. 

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PS4—Final version