MATH 333: Partial Differential Equations

Problem Set 10, Final version

Due Date: Mon., Dec. 7, 2009

Read (or re-read) Sections 10.1–10.4 and Section 4.4 of the Olver text.

$\underline{\star 24}$ Consider the *ordinary* differential equation/BVP (a 1-dimensional version of the Poisson problem)

$$-y''(x) = f(x), \qquad \text{subject to BCs} \qquad y(0) = 0, \quad y'(1) = 1. \tag{1}$$

Let us subdivide the interval $[0, 1]$ into n equally-spaced subintervals,

$$0 = x_0 < x_1 < \ldots < x_n = 1, \qquad \text{with each} \qquad x_m - x_{m-1} = h := \frac{1}{n}.$$

(a) If we use a centered difference approximation for $y''(x)$ then, writing y_m as an approximation to the value of $y(x_m)$, derive the finite difference equation

$$-y_{m-1} + 2y_m - y_{m+1} \;=\; h^2 f(x_m)\,, \tag{2}$$

holding for $m = 1, 2, \ldots, n-1$.

(b) While equation (2) provides most of the particulars for a finite difference *scheme* to approximate the solution of Problem (1), we will not have a *complete* scheme until we specify how to handle the Neumann boundary condition at $x = 1$. Consider two different schemes, based on the following two ways of dealing with this Neumann condition. In the simpler of the two schemes S_1, we approximate $y'(1)$ using a backward difference:

$$\frac{y_n - y_{n-1}}{h} \;=\; 1\,.$$

In scheme S_2, we

- imagine there to be a *ghost node* $x_{n+1} = 1 + h$, outside the right end point (where the Neumann condition occurs) of our interval $[0, 1]$,
- require equation (2) to hold at the right end point x_n (i.e., to hold with $m = n$, which gives rise to an equation involving the ghost node x_{n+1}), and
- introduce the constraint (approximate equation for $y'(1) = 1$)

$$\frac{y_{n+1} - y_{n-1}}{2h} \;=\; 1\,.$$

Schemes S_1 and S_2 both may be turned into matrix problems $\mathbf{A}_1\mathbf{v}_1 = \mathbf{b}_1$ and $\mathbf{A}_2\mathbf{v}_2 = \mathbf{b}_2$, where $\mathbf{v}_1 = (y_1, y_2, \ldots, y_n)$ and $\mathbf{v}_2 = (y_1, y_2, \ldots, y_n, y_{n+1})$ represent the vector of unknowns. Determine the entries of the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and the right-hand sides $\mathbf{b}_1$, $\mathbf{b}_2$.

{c} **This problem part is optional.** Are the matrices $\mathbf{A}_1$, $\mathbf{A}_2$ symmetric? positive definite? (See Problem Set 5, Problem ⋆13, for a previous reference to these concepts. The actual definition of "**A** is positive definite" is that $\mathbf{v}^T\mathbf{A}\mathbf{v} \geq 0$ for all vectors $\mathbf{v}$ of appropriate size, with equality only when $\mathbf{v} = \mathbf{0}$.) If the answers are not obvious, then you may resort to use of numerical evidence. Be clear about what you know to be true (from numerical evidence), and what you only guess at being true. Note: Positive definite matrices are always nonsingular, making a problem like $\mathbf{A}\mathbf{v} = \mathbf{b}$ uniquely solvable.

(d) Let $f(x) = -e^{x-1}$, and show that the resulting true solution of (1) is $y(x) = e^{-1}(e^x - 1)$. Write algorithms to compute finite difference solutions (with this f) both for scheme S_1 and S_2. Compute the maximum error between true and approximate solutions (i.e, maximum difference $|y(x_m) - y_m|$ taken over all grid points $x = x_m$) using both schemes at several different choices of h (say, $h = 0.1$ and $h = 0.05$, at the least). What can you say about the order of convergence for each scheme? (Recall this Excel spreadsheet from the Nov. 13 class that we used to investigate the order of convergence of various finite difference approximations for $f'(x)$.)

<u>10.2.7</u> How would you modify our finite difference schemes for PDEs in order to deal with Neumann boundary conditions? In particular, focus on the *implicit* scheme for solving the heat problem

$$u_t = u_{xx}, \quad 0 < x < 1, \quad t > 0, \quad \text{subject to} \quad \begin{cases} \text{BCs:} & u_x(t,0) = 0 = u_x(t,1), \\ \text{IC:} & u(0,x) = x(x-1/4)(x-2/3). \end{cases}$$

Derive a matrix problem $\mathbf{A}\mathbf{u}^{(j+1)} = \mathbf{b}^{(j+1)}$ to solve, indicating the entries in both $\mathbf{A}$ and $\mathbf{b}^{(j+1)}$, and implement a routine that solves it. If

$$H(t) := \int_0^1 u(t,x)\,dx$$

is a measure of the total heat at time t, then for homogeneous Neumann conditions this total heat should not vary with time (i.e., $H(t)$ should be constant). How closely does your numerical solution maintain this constancy of total heat?

10.4.1 Suppose you wish to numerically solve the IBVP

$$u_{tt} = 8u_{xx},\ \ 0 < x < 3,\ \ t > 0, \quad \text{subject to} \quad \begin{array}{rl} \text{BCs:} & u(t,0) = 0 = u(t,3), \\ \text{ICs:} & u(0,x) = \begin{cases} 1 - 2|x-1|, & \frac{1}{2} \le x \le \frac{3}{2}, \\ 0, & \text{otherwise,} \end{cases} \\ & u_t(0,x) = 0, \end{array}$$

using the finite difference algorithm given in class (which appears as equations (10.61) and (10.57)—or, equivalently, (10.60) and (10.54)—of our text) with spatial step size $\Delta x = 0.1$. What range of time steps Δt are allowed? Test your answer by implementing the numerical solution for one value of Δt in the allowable range and one value outside. Discuss what you observe in your numerical solutions. In the stable range, compare your numerical solution with that obtained with the smaller (spatial) step size $\Delta x = 0.005$.

Also, investigate Exercise 10.4.2 (no work for it need be submitted) to the point of seeing whether any real differences arise in its solution.

4.4.3 Consider the partial differential equation

$$yu_{xx} + (x+y)u_{yy} = 0 .$$

At what points of the plane is the equation elliptic? hyperbolic? parabolic? degenerate?

4.4.5 Steady flow past an airplane is modeled by the partial differential equation

$$(m^2 - 1)u_{xx} + u_{yy} = 0 ,$$

in which x is the flight direction, y the transverse direction, and $m \ge 0$ is the *Mach number*—the ratio of the airplane's speed to the speed of sound. Show that the equation is hyperbolic for subsonic flight, but elliptic for supersonic flight.