

# MATH 333: Partial Differential Equations

## Problem Set 9, Final version

Due Date: Mon., Nov. 16, 2009

Read (at a non-technical level) Sections 11.1–11.2, 11.4 and 11.6 of the Olver text. Section 11.1 contains a subsection entitled “Derivation of the Diffusion Equation.” Even if you have not had vector calculus (of the type in MATH 261/271 or MATH 232 so, in particular, covering *Green’s Theorem*), you should have understood enough of our prior derivation to understand the big picture here on physical grounds. Once the paragraph that contains Equation (11.13) begins, however, the remainder of this subsection relies on material from Chapter 9 which has not been discussed. Section 11.3 is optional, as it tries to give you the material (found, for instance, in Chapter 5 of the Boyce & DiPrima text for MATH 231) for solving linear ODEs with non-constant coefficients (including Bessel’s equation).

★22 Find the solution to

$$u_{tt} - c^2 u_{xx} = f(t, x), \quad 0 < x < \pi, \quad t > 0, \quad \text{subject to} \quad \begin{cases} \text{BCs: } u(t, 0) = 0 = u(t, \pi), \\ \text{ICs: } u(0, x) = 0 = u_t(0, x). \end{cases}$$

Hint: You will have to solve a 2<sup>nd</sup> order inhomogeneous ODE. Recall from MATH 231 that a particular solution  $y_p(t)$  to the ODE

$$y'' + p(t)y' + q(t)y = f(t)$$

is given by the **variation of parameters** formula

$$y_p(t) = \int_0^t \frac{y_1(\tau)y_2(t) - y_2(\tau)y_1(t)}{y_1(\tau)y_2'(\tau) - y_2(\tau)y_1'(\tau)} f(\tau) d\tau,$$

where  $y_1, y_2$  are two linearly independent solutions of the homogeneous equation  $y'' + p(t)y' + q(t)y = 0$ .

As benchmarks along the way, if you assume that  $u(t, x)$  and  $f(t, x)$  have eigenfunction expansions

$$u(t, x) = \sum_{n=1}^{\infty} u_n(t) \sin(nx) \quad \text{and} \quad f(t, x) = \sum_{n=1}^{\infty} f_n(t) \sin(nx),$$

then you should be able to deduce the relationship

$$u_n(t) = A_n \cos(cnt) + B_n \sin(cnt) + \frac{1}{nc} \int_0^t \sin(nc(t - \tau)) f_n(\tau) d\tau.$$

Applying the initial conditions should allow you to conclude that each  $A_n = B_n = 0$ . Then through algebraic manipulation, you should be able to show

$$\begin{aligned} u(t, x) &= \int_0^t \int_0^\pi f(\tau, \xi) \sum_{n=1}^{\infty} \frac{2}{nc\pi} \sin(nc(t - \tau)) \sin(n\xi) \sin(nx) d\xi d\tau \\ &= \int_0^t \langle f(\tau, \cdot), G(t - \tau, x, \cdot) \rangle d\tau, \end{aligned}$$

where  $\langle \cdot, \cdot \rangle$  is the inner product of  $L^2(0, \pi)$ , and

$$G(t, x, \xi) := \sum_{n=1}^{\infty} \frac{2}{nc\pi} \sin(nct) \sin(n\xi) \sin(nx).$$

**★23** Consider again the one-dimensional wave problem of the previous problem. Suppose our forcing function is periodic, having the form  $f(t, x) = \cos(\omega t) \sin(kx)$ , where  $k$  is a positive integer.

- Calculate the inner product  $\langle f(\tau, \cdot), G(t - \tau, x, \cdot) \rangle$ , where  $G(t, x, \xi)$  is as defined in the previous problem.
- Suppose  $\omega \neq ck$ , where  $k$  is the same as appearing in our special forcing function  $f$ , and  $c$  the constant (*wave speed*) in the PDE. Calculate, perhaps using software (a computer algebra system like MATHEMATICA should work here; perhaps even the [online Mathematica integrator](#)), an expression for

$$u(t, x) = \int_0^t \langle f(\tau, \cdot), G(t - \tau, x, \cdot) \rangle d\tau.$$

This  $u(t, x)$  should actually be the sum of *two* expressions, one which satisfies the *unforced* (i.e., homogeneous) wave equation, and the other which satisfies the forced one. Identify this latter part, and compare it to the forcing function  $f(t, x)$  itself.

- Calculate an expression for  $u(t, x)$  again, this time assuming that  $\omega = ck$ . What phenomenon does this solution illustrate, and why?
- 11.4.3**
- Write down the first three nonzero terms in the Fourier-Bessel series for the solution to the heat equation on the unit disk (see Section 11.4 in Olver)  $\mathbb{D} = \{(r, \theta) \mid 0 \leq r < 1, -\pi \leq \theta < \pi\}$  with  $\gamma = 1$ , whose circular edge is held at  $0^\circ$  subject to the IC  $u(0, x, y) \equiv 1$  for  $x^2 + y^2 \leq 1$ . Use numerical integration to evaluate the coefficients of these terms.
  - Use your approximation to determine at which times  $t \geq 0$  the temperature of the disk is everywhere  $\leq 0.5$ .

11.4.18 Demonstrate directly that the eigenfunctions

$$\phi_{mn}(r, \theta) = J_m(\alpha_{m,n}r) \cos(m\theta) \quad \text{and} \quad \tilde{\phi}_{mn}(r, \theta) = J_m(\alpha_{m,n}r) \sin(m\theta)$$

of the Laplacian operator on the unit disk  $\mathbb{D} = \{(r, \theta) \mid 0 \leq r < 1, -\pi \leq \theta < \pi\}$  subject to periodic BCs in  $\theta$  and homogeneous Dirichlet BC in  $r$  are orthogonal with respect to the  $L^2$  inner product on  $\mathbb{D}$ .