MATH 333: Partial Differential Equations

Problem Set 8, Final version

Due Date: Mon., Nov. 9, 2009

4.3.29 [This problem gives a partial answer to the question of what forms *harmonic functions* of 2 variables can take.] Find an harmonic function $u(x, y)$ defined on the annulus $\frac{1}{2} < \|\mathbf{x}\| < 1$ (where $\mathbf{x} = (x, y)$) subject to the constant Dirichlet BCs $u = a$ on $\|\mathbf{x}\| = \frac{1}{2}$ and $u = b$ on $\|\mathbf{x}\| = 1$. You may assume, as seems physically reasonable given the BCs, that the solution is radially symmetric, so that $u = u(r)$ is a function of $r = \sqrt{x^2 + y^2}$ alone (independent of θ), and so Laplace's equation reduces to

$$\Delta u \;=\; \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \;=\; 0\,.$$

Hint: Let $v = u_r$ and solve the 1^{st} order ODE for v. Then find u.

8.3.7 Consider the heat problem

$$w_t = \gamma w_{xx}, \quad 0 < x < \ell, \quad t > 0, \quad \text{subject to} \quad \begin{cases} \text{Dirichlet BCs:} & w(t,0) = 0, \\ & w(t,\ell) = 0, \\ \text{IC:} & w(0,x) = \varphi(x). \end{cases} \tag{1}$$

(a) Suppose w to be a non-constant solution of (1) augmented with homogeneous Dirichlet BCs, and define

$$E(t) \;:=\; \int_0^\ell w^2(t,x)\,dx\,.$$

Show that $E(\cdot)$ is a non-increasing function. Hint: Differentiate E, pass the derivative through the integral, and use the fact that w solves the heat equation.

(b) Consider now the *forced* heat equation with non-homogeneous Dirichlet BCs

$$u_t = \gamma u_{xx} + f(t,x), \quad 0 < x < \ell, \quad t > 0, \quad \text{subject to} \quad \begin{cases} \text{Dirichlet BCs:} & u(t,0) = \alpha(t), \\ & u(t,\ell) = \beta(t), \\ \text{IC:} & u(0,x) = \varphi(x). \end{cases} \tag{2}$$

Use the result from part (a) to demonstrate that (2) can have at most one solution.

(c) Would you be able to draw the same *uniqueness* conclusion if the boundary conditions in (2) were of

- Neumann type: $u_x(t,0) = \alpha(t), \quad u_x(t,\ell) = \beta(t)$?
- mixed type: $u(t,0) = \alpha(t), \quad u_x(t,\ell) = \beta(t)$?

Why or why not?

★20 Show that the Poisson problem with Dirichlet BCs on a bounded domain Ω is stable. Specifically, suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded and connected, and consider the two (nearly identical—they both have the same forcing function f) problems

$$\Delta u(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad \text{subject to} \qquad u(\mathbf{x}) = g_1(\mathbf{x}), \ \text{ for } \ \mathbf{x} \in \partial\Omega, \tag{3}$$

and

$$\Delta v(\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \Omega, \qquad \text{subject to} \qquad v(\mathbf{x}) = g_2(\mathbf{x}), \ \text{ for } \ \mathbf{x} \in \partial\Omega. \tag{4}$$

If boundary data for the two problems are similar—that is, if

$$\|g_1 - g_2\|_\infty \ := \ \max\{|g_1(\mathbf{x}) - g_2(\mathbf{x})| \ : \ \mathbf{x} \in \partial\Omega\}$$

is small—show that the difference in solutions is small throughout Ω.

★21 Consider the following BVP:

$$\begin{aligned} \Delta u &= 0, \qquad (x,y) \in \Omega \\ u(x,y) &= x^2 + y^2 - 5x, \qquad (x,y) \in \partial\Omega, \end{aligned}$$

where Ω is the closed region in the xy-plane bounded by the circle centered at the origin of radius 2: $x^2 + y^2 = 4$. What are the maximum and minimum values of u and at what points do these values occur?