MATH 333: Partial Differential Equations

Problem Set 7, Final version

Due Date: Mon., Nov. 2, 2009

Read Sections 4.3–4.4, 8.3 of the Olver text.

★17 In this exercise we explore the properties of a complete orthonormal system on $(-\infty, \infty)$ called the Haar wavelets. Let ϕ be the function (called a *scaling function*)

$$\phi(x) \;=\; \begin{cases} 1, & \text{if } x \in [0,1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(x) = \phi(2x) - \phi(2x-1)$. (Some people refer to ψ as the *mother wavelet*, but others appear to give that name to the scaling function.) The Haar wavelets are the (doubly-subscripted) functions

$$\psi_{mn}(x) \;:=\; 2^{m/2}\psi(2^m x - n), \qquad m, n = 0, \pm 1, \pm 2, \ldots.$$

(a) Sketch graphs (you may do so by hand if you like) for ψ and ψ_{mn}, m, $n = 0$, ± 1, ± 2 (representatives from this range of m, n), and give a piecewise-defined formula (one that does not explicitly refer to the function ϕ) for ψ. Deduce that

$$\psi_{mn}(x) \;=\; \begin{cases} 2^{m/2}, & \text{if } n2^{-m} \le x < (n+1/2)2^{-m}, \\ -2^{m/2}, & \text{if } (n+1/2)2^{-m} \le x < (n+1)2^{-m}, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Prove that the ψ_{mn} are mutually orthogonal and that they are normalized (i.e., that $\|\psi_{mn}\|_{L^2(\mathbb{R})} = 1$ for each m, n). Note that the word *orthonormal* means *orthogonal* + *normal*. Hint: First convince yourself that, for $m \ge k$,

$$\psi_{mn}(x)\psi_{k\ell}(x) \;=\; \begin{cases} \psi^2_{mn}(x), & \text{if } m = k, \; n = \ell, \\ 2^{k/2}\,\psi_{mn}(x), & \text{if } m > k, \; n = \ell \cdot 2^{m-k}, 1 + \ell \cdot 2^{m-k}, \ldots, (\ell + 1/2)\cdot 2^{m-k} - 1, \\ -2^{k/2}\,\psi_{mn}(x), & \text{if } m > k, \; n = \frac{2\ell+1}{2}\cdot 2^{m-k}, \frac{2\ell+3}{2}\cdot 2^{m-k}, \ldots, (\ell+1)\cdot 2^{m-k} - 1, \\ 0, & \text{otherwise.} \end{cases}$$

(c) The completeness of the Haar wavelets means that every $f \in L^2(\mathbb{R})$ has a generalized Fourier series expansion

$$f(x) \;=\; \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn}\psi_{mn}(x).$$

Find a formula (integral form, but simplified as much as possible) for the coefficients c_{mn}.

{$\star$18} {Optional exercise, but suggested particularly for those interested in Schrödinger's equation}

Hermite functions. Consider the differential equation

$$-y'' + x^2 y \;=\; Ey, \quad x \in \mathbb{R}, \quad E = \text{constant},$$

and the functions

$$H_n(x) \;=\; (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \qquad n = 0, 1, 2, \ldots.$$

(a) Find $H_n(x)$, for $n = 0, 1, \ldots, 4$. Why is $H_n(x)$ always a polynomial (known as an *Hermite polynomial*)?

(b) Verify that $v_n(x) = H_n(x) e^{-x^2/2}$ is a solution of the differential equation when $E = 2n + 1$. Hint: First, note that

$$\begin{aligned}
\frac{d^{n+1}}{dx^{n+1}}(e^{-x^2}) &= \frac{d^n}{dx^n}(-2xe^{-x^2}) \\
&= (-2x)\frac{d^n}{dx^n}(e^{-x^2}) + \binom{n}{1}\frac{d}{dx}(-2x)\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2}) \\
&\quad + \left\{\text{terms of form } \frac{d^k}{dx^k}(-2x)\frac{d^{n-k}}{dx^{n-k}}, \text{ where } k \geq 2\right\} \\
&= -2x\frac{d^n}{dx^n}(e^{-x^2}) - 2n\frac{d^{n-1}}{dx^{n-1}}(e^{-x^2}),
\end{aligned}$$

and use this to show $H_n' = 2nH_{n-1}$.

(c) Show that $\int_{-\infty}^{\infty} v_n v_m \, dx = 0$, $m \neq n$, and thus the v_n are orthogonal on the interval $(-\infty, \infty)$. Hint: Use the fact that both v_n and v_m satisfy the differential equation for appropriate E to show that $v_m'' v_n - v_n'' v_m = 2(n - m) v_m v_n$.

(d) If a function $f(x)$ can be represented by $f(x) = \sum_{n=0}^{\infty} c_n v_n(x)$, how would you expect to find the c_n? Assume uniform convergence. Take $f(x) = 1/\sqrt{1 + x^4}$ and use software to find c_n, $n = 0, \ldots 4$.

$\star$19 Suppose $A = \{\varphi_1, \varphi_2, \ldots\}$ is an orthogonal collection of functions in $L^2(a, b)$. An important result known as **Bessel's Inequality** states that, for any $f \in L^2(a, b)$, we have

$$\|f\|_2^2 \;\geq\; \sum_j \frac{\left|\left\langle f, \varphi_j \right\rangle\right|^2}{\|\varphi_j\|_2^2}\,. \tag{1}$$

Note that the sum on the right-hand side is finite or infinite depending on the number of functions ϕ_j found in S. Consider the series solution

$$u(t, x) \;=\; \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right), \qquad \text{with} \qquad c_n \;=\; \frac{\langle f, \sin(n\pi \cdot /\ell)\rangle}{\|\sin(n\pi \cdot /\ell)\|_2^2}\,,$$

of the heat problem with Dirichlet BCs

$$u_t = u_{xx}, \quad 0 < x < \ell, \quad t > 0, \quad \text{subject to} \quad \begin{cases} \text{BCs:} & u(t,0) = 0 = u(t,\ell), \\ \text{IC:} & u(0,x) = f(x). \end{cases}$$

In this problem we investigate the error incurred by truncating the series solution.

(a) Give a geometric explanation of why Bessel's Inequality (1) should be true. Hint: Consider the subspace $\mathcal{S}$ of $L^2(a,b)$ spanned by the collection A. Write out an expression for $\|v\|_2^2$, where $v = \text{proj}_{\mathcal{S}} f$.

(b) Suppose N is a fixed positive integer and we define

$$T_N(t,x) \;:=\; \sum_{n=1}^{N} c_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right),$$

and a remainder

$$R_N(t,x) \;:=\; u(t,x) - T_N(t,x) \;=\; \sum_{n=N+1}^{\infty} c_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi x}{\ell}\right),$$

so that $u(t,x) = T_N(t,x) + R_N(t,x)$ is the sum of the series truncated at the N^{th} term and this remainder. At any fixed $t > 0$, the square of the 2-norm of R_N is defined to be

$$\begin{aligned} \|R_N(t,\cdot)\|_2^2 &= \left\| \sum_{n=N+1}^{\infty} c_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi\cdot}{\ell}\right) \right\|_2^2 \\ &= \left\langle \sum_{n=N+1}^{\infty} c_n e^{-n^2\pi^2 t/\ell^2} \sin\left(\frac{n\pi\cdot}{\ell}\right), \sum_{k=N+1}^{\infty} c_k e^{-k^2\pi^2 t/\ell^2} \sin\left(\frac{k\pi\cdot}{\ell}\right) \right\rangle \end{aligned}$$

Employ the orthogonality of the functions $\{\sin(n\pi\cdot/\ell)\}_{n=1}^{\infty}$ in $L^2(0,\ell)$ to simplify this expression for $\|R_N(t,\cdot)\|_2^2$, and then use Bessel's inequality to conclude that, for $t \geq t_0 > 0$,

$$\|R_N(t,\cdot)\|_2 \;\leq\; e^{-(N+1)^2\pi^2 t_0/\ell^2} \|f\|_2 \,.$$

(c) Now take $\ell = \pi$ and

$$f(x) \;=\; \begin{cases} 0, & 0 < x < \pi/2, \\ 1, & \pi/2 < x < \pi. \end{cases}$$

If we want the 2-norm of the remainder $\|R_N(t,\cdot)\|_2 < 0.01$ for all $t \geq 0.1$, use your estimate from part (b) to determine a sufficiently large N. That is, determine N so that the truncated series $T_N(t,x)$ approximates $u(t,x)$ this accurately (in 2-norm) whenever $t \geq 0.1$. For this N, produce plots on $[0,\pi]$ for $f(x)$, $T_N(0,x)$, $T_N(0.1,x)$ and $T_N(0.2,x)$.