MATH 333: Partial Differential Equations

Problem Set 6, Final version

Due Date: Fri., Oct. 23, 2009

Read Sections 4.1–4.2 of the Olver text.

$\star$14 No doubt you have dealt more with real-valued than with complex-valued functions in your academic life. Even if a function accepts only real inputs, its outputs may be nonreal, as in

$$f(x) \;=\; u(x) + iv(x)\;.$$

Differentiation and integration of functions is done on the real and imaginary parts separately:

$$\frac{d}{dx}f(x) \;=\; \frac{d}{dx}u(x) + i\frac{d}{dx}v(x) \;=\; u'(x) + iv'(x)\;,$$

and

$$\int_a^b f(x)\,dx \;=\; \int_a^b u(x) + i\int_a^b v(x)\,dx\;.$$

Recall that the $L^2(a,b)$ inner product between functions f and g is defined by

$$\langle f, g\rangle \;:=\; \int_a^b f(x)\overline{g(x)}\,dx\;. \tag{1}$$

(a) Show that, given a complex scalar α, the inner product (1) satisfies

$$\langle \alpha f, g\rangle \;=\; \alpha\,\langle f, g\rangle \qquad \text{and} \qquad \langle f, \alpha g\rangle \;=\; \overline{\alpha}\,\langle f, g\rangle\;.$$

(b) In class we developed the complex exponential form of the Fourier series of $f \in L^2(-\ell, \ell)$:

$$f(x) \;=\; \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{in\pi x}{\ell}\right), \qquad \text{where} \qquad c_n \;=\; \frac{\left\langle f, \exp\left(\frac{in\pi\cdot}{\ell}\right)\right\rangle}{\left\|\exp\left(\frac{in\pi\cdot}{\ell}\right)\right\|_2^2}\;. \tag{2}$$

Find, for each $n = \ldots, -2, -1, 0, 1, \ldots$, the value of $\|\exp(in\pi\cdot/\ell)\|_2^2$. Does your answer depend upon n?

(c) In class we concluded that, given the expression for the c_n's in (2), we have for each integer $n > 0$,

$$c_{-n}\exp\left(\frac{i(-n)\pi x}{\ell}\right) + c_n\exp\left(\frac{in\pi x}{\ell}\right) \;=\; a_n\cos\left(\frac{n\pi x}{\ell}\right) + b_n\sin\left(\frac{n\pi x}{\ell}\right), \tag{3}$$

where the a_n's, b_n's are given by the usual expressions

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \qquad \text{and} \qquad b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx\,. \quad (4)$$

Use equations (2), (3) and (4) to establish the relationships (3.66) from p. 88 of the text. [Hint: Equate the sin() and cos() parts of (3).]

4.1.2 Consider the IBVP

$$u_t = u_{xx}, \quad 0 < x < 10, \quad t > 0, \qquad \text{subject to BCs:} \quad u(t,0) = 0 = u(t,10),$$

and

$$\text{IC:} \quad u(0,x) = f(x) := \begin{cases} x-1, & 1 \le x \le 2, \\ 11-5x, & 2 \le x \le 3, \\ 5x-19, & 3 \le x \le 4, \\ 5-x, & 4 \le x \le 5, \\ 0, \text{otherwise}. \end{cases}$$

Discuss what happens to the solution as t increases. You do *not* need to write down an explicit formula for $u(t,x)$ (in particular, you do not need to give the paper calculations of the Fourier coefficients), but for full credit you must explain (sketches can help) at least 3–4 interesting things that happen to the solution as time progresses. The OCTAVE file `fourierCoeffs.m` may well be of use to you here, as well as some modified version of the file `fss_approx.m`. An OCTAVE version of the function f which carries out pointwise evaluation on vectors is

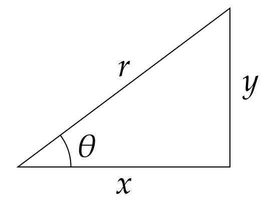
```
function y = f(x)
  y = zeros(size(x));
  y = (x >= 1 & x <= 2) .* (x - 1);
  y += (x > 2 & x <= 3) .* (11 - 5*x);
  y += (x > 3 & x <= 4) .* (5*x - 19);
  y += (x > 4 & x <= 5) .* (5 - x);
end
```

⋆15 In class we showed that the Laplacian with homogeneous Dirichlet BCs is self-adjoint. Our argument assumed we were working in three dimensions. Provide an argument (you may, again, work in three dimensions) that shows the Laplacian with homogeneous Neumann conditions is self-adjoint. Note that, in one spatial dimension, homogeneous Neumann conditions look like

$$u_x(t,a) = 0 = u_x(t,b)\,,$$

which says that there is zero flow past the boundaries of the interval (a, b). You must determine what the analogous condition of "zero flow across the boundary" would look like when the region Ω is an open and connected subset of $\mathbb{R}^3$.

$\star$16 Rectangular coordinates (x, y) and polar coordinates (r, θ) have the following relationships:



$$x = r\cos\theta \qquad r = \sqrt{x^2 + y^2}$$
$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$

(a) The chain rule says $\partial u/\partial x = (\partial u/\partial r)(\partial r/\partial x) + (\partial u/\partial\theta)(\partial\theta/\partial x)$. Use the chain rule along with the relationships above to conclude that

$$\frac{\partial}{\partial x} = \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\,.$$

(b) Working as in part (a), it is possible to conclude

$$\frac{\partial}{\partial y} = \sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\,.$$

Use this and the result from (a) to write the two-dimensional Laplacian operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ in polar coordinates. (Note: The polar expressions for $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ are a good deal uglier than that for Δ; there are a lot of cancellations when the two are combined.)