MATH 333: Partial Differential Equations

Problem Set 5, Final version

Due Date: Mon., Oct. 12, 2009

Carefully read back through Examples 3.3 and 3.14 in Section 3.2 of the Olver text. Read Sections 3.3–3.4; mathematics majors should consider reading Section 3.5 as well. Most of the discussion in 3.3 is devoted to integrating Fourier series; read that mostly for background, and focus your reading when you get to the last half-page of the section devoted to differenting Fourier series.

$\star$12 In the subsection titled "The Heated Ring" (p. 66 ff), our textbook's author takes up the solution of the heat problem

$$u_t = u_{xx}, \quad -\pi \le x \le \pi, \quad t > 0, \qquad \text{subject to} \qquad \begin{cases} u(0,x) = f(x), \\ u(t,-\pi) = u(t,\pi), \\ u_x(t,-\pi) = u_x(t,\pi)\ . \end{cases}$$

(These BCs here are called *periodic boundary conditions*.) For a fixed $\ell > 0$, consider the more general problem

$$u_t = u_{xx}, \quad -\ell \le x \le \ell, \quad t > 0, \qquad \text{subject to} \qquad \begin{cases} u(0,x) = f(x), \\ u(t,-\ell) = u(t,\ell), \\ u_x(t,-\ell) = u_x(t,\ell)\ . \end{cases} \tag{1}$$

(a) Assuming separable solutions $u(t,x) = e^{\lambda t}v(x)$, one arrives at the eigenvalue problem (akin to equation (3.23) in the text)

$$v'' = \lambda v\ , \qquad v(-\ell) = v(\ell)\ , \qquad v'(-\ell) = v'(\ell)\ ,$$

at which point, the author looks at three possibilities: $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. While he concludes there are no eigenvalues $\lambda > 0$, he provides no argument for why eigenvalues cannot be (non-real) complex. Give an alternate argument which both rules out positive *and* non-real eigenvalues for the problem (stated on $-\ell < x < \ell$) above.

(b) For his version of the heated ring problem, Olver says that each negative eigenvalue "admits a two-dimensional space of eigenfunctions." This is true for problem (1) above as well which, for a given negative eigenvalue $\lambda_n = -n^2\pi^2/\ell$, $n = 1, 2, \ldots$, has two independent eigenfunctions

$$v_n(x) \ = \ \cos\left(\frac{n\pi x}{\ell}\right) \qquad \text{and} \qquad \tilde{v}_n(x) \ = \ \sin\left(\frac{n\pi x}{\ell}\right)\ .$$

Show that, under the $L^2(-\ell,\ell)$ inner product, $v_n(\cdot)$ and $\tilde{v}_n(\cdot)$ are orthogonal (no matter which *nonzero* n we use).

(c) The claim that each $f \in L^2(-\ell,\ell)$ has a *classical Fourier series* expansion

$$f \ \sim \ \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{\ell}\right) + b_n \sin\left(\frac{n\pi x}{\ell}\right) \right] ,$$

with

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\left(\frac{n\pi x}{\ell}\right) \qquad \text{and} \qquad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\left(\frac{n\pi x}{\ell}\right)$$

relies on the fact that the eigenfunctions

$$S = \{1, \cos(\pi x/\ell), \sin(\pi x/\ell), \cos(2\pi x/\ell), \sin(2\pi x/\ell), \ldots\}$$

form a *basis* for $L^2(-\ell,\ell)$. This seems odd, given that the vector space $L^2(0,\ell)$ has two bases (we know of) which are proper subsets of S—namely

$$\{1, \cos(\pi x/\ell), \cos(2\pi x/\ell), \ldots\} \qquad \text{and} \qquad \{\sin(\pi x/\ell), \sin(2\pi x/\ell), \ldots\}.$$

Demonstrate, perhaps using software, that functions from the basis on the right, when considered as elements in $L^2(0,\ell)$, have Fourier cosine series expansions (i.e., expansions in terms of the basis on the left). Show, further, that this is no longer the case when the space is enlarged to $L^2(-\ell,\ell)$.

There are several programs I have used for demonstrating the convergence of Fourier series in class. These are `fss_approx.m`, `fcs_approx.m`, and `cfs_approx.m`, as well as the program `fourierCoeffs.m`, on which all the others rely. All of these are located in the directory `http://www.calvin.edu/~scofield/courses/m333/materials/octave/`.

★13 (a) Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$. Prove the following statement: An n-by-n matrix $\mathbf{A}$ is symmetric if and only if $\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ay} \rangle$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Recall that, by definition, $\mathbf{A}$ is symmetric precisely when $\mathbf{A} = \mathbf{A}^T$.

Hint: Use the fact that $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T \mathbf{x}$ where the right-hand side is matrix multiplication between the 1-by-n vector $\mathbf{y}^T$ and the n-by-1 vector $\mathbf{x}$. Also, recall that for a general matrix product, $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$.

(b) Let L be a linear differential operator defined on some class C of functions in $L^2(a,b)$. The operator L is said to be **self-adjoint** (or **Hermitian**) if $\langle L[u], v \rangle = \langle u, L[v] \rangle$ for each $u, v \in C$. (Now $\langle \cdot, \cdot \rangle$ stands for the inner product in $L^2(a,b)$.)

Given the similarity in requirements, it is reasonable that self-adjoint operators are counterparts to symmetric matrices.

As an example, consider the linear differential operator $L[\phi] = \phi''$, defined for functions $\phi \in C^2([a, b])$ that satisfy the BCs $\phi(a) = 0$, $\phi(b) = 0$. For any other function ψ in $C^2([a, b])$ satisfying the same boundary conditions, we have

$$\begin{aligned}
\left\langle L[\phi], \psi \right\rangle &= \int_a^b \phi''(x)\overline{\psi(x)}\,dx = \phi'(x)\overline{\psi(x)}\Big|_a^b - \int_a^b \phi'(x)\overline{\psi'(x)}\,dx \\
&= \phi'(b)\overline{\psi(b)} - \phi'(a)\overline{\psi(a)} - \int_a^b \phi'(x)\overline{\psi'(x)}\,dx \\
&= -\int_a^b \phi'(x)\overline{\psi'(x)}\,dx \\
&= -\phi(x)\overline{\psi'(x)}\Big|_a^b + \int_a^b \phi(x)\overline{\psi''(x)}\,dx \\
&= \int_a^b \phi(x)\overline{\psi''(x)}\,dx = \left\langle \phi, L[\psi] \right\rangle ,
\end{aligned}$$

and so L (with these boundary conditions) is self-adjoint.

Consider now the derivative operator $A = d^4/dx^4$ acting on the collection of functions ϕ in $C^4([0, 1])$ which satisfy $\phi(0) = \phi'(0) = 0 = \phi(1) = \phi'(1)$. This operator arises in the Bernoulli-Euler model of transverse vibrations of a beam clamped at both ends. Show that A is self-adjoint.

(c) Show that the operator A coupled with clamped BCs introduced in the previous part is **positive definite**—that is, that its eigenvalues are all positive real numbers.

3.2.1 Find the (classical, taking the interval $-\pi < x \leq \pi$) Fourier series of $\sin^2 x$ and $\cos^2 x$ *without directly calculating* (in contrast to Exercise 3.2.3(a)) the Fourier coefficients. [Hint: Use some standard trigonometric identities. (Note: This is *not* an exercise that calls for the use of software.)

3.2.3 (a) Find the (classical, again working in $-\pi < x \leq \pi$) Fourier series for the function

$$f(x) = \begin{cases} 1, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise.} \end{cases}$$

(Note: This is *not* an exercise calling for the use of software.)

3.2.18 Is $x^{1/3}$ piecewise continuous? piecewise C^1? piecewise C^2?

3.2.26 (a) Sketch the 2π-periodic half wave $f(x) = \begin{cases} \sin x, & 0 < x \le \pi, \\ 0, & -\pi \le x < 0. \end{cases}$

(b) Find its (classical) Fourier series.

(c) Graph the first 5 Fourier sums and compare with the function.

(d) Discuss convergence of the Fourier series.