Carefully read back through Examples 3.3 and 3.14 in Section 3.2 of the Olver text. Read Sections 3.3–3.4; mathematics majors should consider reading Section 3.5 as well. Most of the discussion in 3.3 is devoted to integrating Fourier series; read that mostly for background, and focus your reading when you get to the last half-page of the section devoted to differenting Fourier series.

12 In the subsection titled “The Heated Ring” (p. 66 ff), our textbook’s author takes up the solution of the heat problem

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\pi \leq x \leq \pi, \quad t > 0, \quad \text{subject to} \]

\[
\begin{align*}
u(0, x) &= f(x), \\
u(t, -\pi) &= u(t, \pi), \\
u_x(t, -\pi) &= u_x(t, \pi).
\end{align*}
\]

(These BCs here are called periodic boundary conditions.) For a fixed \( \ell > 0 \), consider the more general problem

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -\ell \leq x \leq \ell, \quad t > 0, \quad \text{subject to} \]

\[
\begin{align*}
u(0, x) &= f(x), \\
u(t, -\ell) &= u(t, \ell), \\
u_x(t, -\ell) &= u_x(t, \ell). \tag{1}
\end{align*}
\]

(a) Assuming separable solutions \( u(t, x) = e^{\lambda t}v(x) \), one arrives at the eigenvalue problem (akin to equation (3.23) in the text)

\[ v'' = \lambda v, \quad v(-\ell) = v(\ell), \quad v'(-\ell) = v'(\ell), \]

at which point, the author looks at three possibilities: \( \lambda > 0 \), \( \lambda = 0 \), and \( \lambda < 0 \). While he concludes there are no eigenvalues \( \lambda > 0 \), he provides no argument for why eigenvalues cannot be (non-real) complex. Give an alternate argument which both rules out positive and non-real eigenvalues for the problem (stated on \(-\ell < x < \ell\)) above.

(b) For his version of the heated ring problem, Olver says that each negative eigenvalue “admits a two-dimensional space of eigenfunctions.” This is true for problem (1) above as well which, for a given negative eigenvalue \( \lambda_n = -n^2 \pi^2 / \ell \), \( n = 1, 2, \ldots \), has two independent eigenfunctions

\[ v_n(x) = \cos \left( \frac{n\pi x}{\ell} \right) \quad \text{and} \quad \tilde{v}_n(x) = \sin \left( \frac{n\pi x}{\ell} \right). \]
Show that, under the \( L^2(-\ell, \ell) \) inner product, \( v_n(\cdot) \) and \( \tilde{v}_n(\cdot) \) are orthogonal (no matter which nonzero \( n \) we use).

(c) The claim that each \( f \in L^2(-\ell, \ell) \) has a **classical Fourier series** expansion

\[
f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[ a_n \cos \left( \frac{n\pi x}{\ell} \right) + b_n \sin \left( \frac{n\pi x}{\ell} \right) \right],
\]

with

\[
a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos \left( \frac{n\pi x}{\ell} \right) \quad \text{and} \quad b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin \left( \frac{n\pi x}{\ell} \right)
\]

relies on the fact that the eigenfunctions

\[
S = \{1, \cos(\pi x/\ell), \sin(\pi x/\ell), \cos(2\pi x/\ell), \sin(2\pi x/\ell), \ldots\}
\]

form a **basis** for \( L^2(-\ell, \ell) \). This seems odd, given that the vector space \( L^2(0, \ell) \) has two bases (we know of) which are proper subsets of \( S \)—namely

\[
\{1, \cos(\pi x/\ell), \cos(2\pi x/\ell), \ldots\} \quad \text{and} \quad \{\sin(\pi x/\ell), \sin(2\pi x/\ell), \ldots\}.
\]

Demonstrate, perhaps using software, that functions from the basis on the right, when considered as elements in \( L^2(0, \ell) \), have Fourier cosine series expansions (i.e., expansions in terms of the basis on the left). Show, further, that this is no longer the case when the space is enlarged to \( L^2(-\ell, \ell) \).

There are several programs I have used for demonstrating the convergence of Fourier series in class. These are \texttt{fss_approx.m}, \texttt{fcs_approx.m}, and \texttt{cfs_approx.m}, as well as the program \texttt{fourierCoeffs.m}, on which all the others rely. All of these are located in the directory \texttt{http://www.calvin.edu/~scofield/courses/m333/materials/octave/}.

\( \star 13 \) (a) Let \( \langle \cdot, \cdot \rangle \) denote the usual inner product on \( \mathbb{R}^n \). Prove the following statement:

An \( n \)-by-\( n \) matrix \( A \) is symmetric if and only if \( \langle Ax, y \rangle = \langle x, Ay \rangle \) for all \( x, y \in \mathbb{R}^n \).

Recall that, by definition, \( A \) is symmetric precisely when \( A = A^T \).

Hint: Use the fact that \( \langle x, y \rangle = y^T x \) where the right-hand side is matrix multiplication between the \( 1 \)-by-\( n \) vector \( y^T \) and the \( n \)-by-1 vector \( x \). Also, recall that for a general matrix product, \( (AB)^T = B^T A^T \).

(b) Let \( L \) be a linear differential operator defined on some class \( C \) of functions in \( L^2(a, b) \). The operator \( L \) is said to be **self-adjoint** (or **Hermitian**) if \( \langle L[u], v \rangle = \langle u, L[v] \rangle \) for each \( u, v \in C \). (Now \( \langle \cdot, \cdot \rangle \) stands for the inner product in \( L^2(a, b) \).)
Given the similarity in requirements, it is reasonable that self-adjoint operators are counterparts to symmetric matrices.

As an example, consider the linear differential operator $L[\phi] = \phi''$, defined for functions $\phi \in C^2([a, b])$ that satisfy the BCs $\phi(a) = 0, \phi(b) = 0$. For any other function $\psi$ in $C^2([a, b])$ satisfying the same boundary conditions, we have

$\langle L[\phi], \psi \rangle = \int_a^b \phi''(x)\overline{\psi(x)} \, dx = \phi'(b)\overline{\psi(b)} - \phi'(a)\overline{\psi(a)} - \int_a^b \phi'(x)\overline{\psi'(x)} \, dx$

$= -\int_a^b \overline{\phi'(x)\psi'(x)} \, dx$

$= -\phi(x)\overline{\psi'(x)} \bigg|_a^b + \int_a^b \phi(x)\overline{\psi''(x)} \, dx$

$= \int_a^b \phi(x)\overline{\psi''(x)} \, dx = \langle \phi, L[\psi] \rangle,$

and so $L$ (with these boundary conditions) is self-adjoint.

Consider now the derivative operator $A = d^4/dx^4$ acting on the collection of functions $\phi$ in $C^4([0, 1])$ which satisfy $\phi(0) = \phi'(0) = 0 = \phi(1) = \phi'(1)$. This operator arises in the Bernoulli-Euler model of transverse vibrations of a beam clamped at both ends. Show that $A$ is self-adjoint.

(c) Show that the operator $A$ coupled with clamped BCs introduced in the previous part is positive definite—that is, that its eigenvalues are all positive real numbers.

3.2.1 Find the (classical, taking the interval $-\pi < x \leq \pi$) Fourier series of $\sin^2 x$ and $\cos^2 x$ without directly calculating (in contrast to Exercise 3.2.3(a)) the Fourier coefficients. [Hint: Use some standard trigonometric identities. (Note: This is not an exercise that calls for the use of software.)

3.2.3 (a) Find the (classical, again working in $-\pi < x \leq \pi$) Fourier series for the function

$f(x) = \begin{cases} 1, & |x| < \frac{1}{2}\pi, \\ 0, & \text{otherwise}. \end{cases}$

(Note: This is not an exercise calling for the use of software.)
3.2.18 Is \( x^{1/3} \) piecewise continuous? piecewise \( C^1 \)? piecewise \( C^2 \)?

3.2.26 (a) Sketch the \( 2\pi \)-periodic half wave \( f(x) = \begin{cases} \sin x, & 0 < x \leq \pi, \\ 0, & -\pi \leq x < 0. \end{cases} \)

(b) Find its (classical) Fourier series.

(c) Graph the first 5 Fourier sums and compare with the function.

(d) Discuss convergence of the Fourier series.