MATH 333: Partial Differential Equations

Problem Set 4, Final version

Due Date: Mon., Oct. 5, 2009

Read Sections 3.1–3.2 from the Olver text. Read Sections 8–9 of the OCTAVE tutorial found at http://www-mdp.eng.cam.ac.uk/web/CD/engapps/octave/octavetut.pdf.

★8 Suppose we wish to solve the 1st-order linear nonhomogeneous Cauchy problem

$$\begin{aligned} &u_t + a(x,t)u_x + b(x,t)u = f(x,t), \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = \phi(x), \quad x \in \mathbb{R}. \end{aligned} \tag{1}$$

Consider the related problems

$$\begin{aligned} &v_t + a(x,t)v_x + b(x,t)v = 0, \quad x \in \mathbb{R},\ t > 0 \\ &v(x,0) = \phi(x), \quad x \in \mathbb{R}, \end{aligned} \tag{2}$$

and

$$\begin{aligned} &w_t + a(x,t)w_x + b(x,t)w = f(x,t), \quad x \in \mathbb{R},\ t > 0 \\ &w(x,0) = 0, \quad x \in \mathbb{R}. \end{aligned} \tag{3}$$

Without attempting to solve problems (2), (3) (i.e., without expressions for their solutions v and w), show that when v and w solve (2), (3) respectively, then $u = v + w$ solves (1). (Don't forget to verify that u satisfies the appropriate initial condition.)

★9 Functions of the form x^r, where r is a (fixed) real number are called "power functions". When $r \geq 0$, the resulting power function is continuous on all of $[0,1]$, and using some of the theorems of Calculus, it is easy to establish that the function is, therefore, a member of $L^2(0,1)$. But what of the power functions with $r < 0$? They are continuous on the open interval $(0,1)$, but not on the closed interval $[0,1]$. Are any of them members of $L^2(0,1)$? For which choices of $r < 0$? Note that, for $r < 0$,

$$\int_0^1 x^{2r}\,dx$$

is an improper integral.

★10 (a) The following expresses *two* equations, the trigonometric identities for the cosine of the sum and difference of angles:

$$\cos(\phi \pm \psi) \;=\; \cos\phi\cos\psi \mp \sin\phi\sin\psi\ .$$

Use these equations to establish another trigonometric identity, namely

$$\sin\phi\sin\psi \;=\; \frac{1}{2}\Big[\cos(\phi-\psi) - \cos(\phi+\psi)\Big]\ .$$

(b) We have defined an inner product on $L^2(a,b)$ to be

$$\langle f, g\rangle \;:=\; \int_a^b f(x)\overline{g(x)}\,dx\ .$$

Consider two sine functions $\sin(n\pi\cdot)$ and $\sin(m\pi\cdot)$, where m, n are both positive integers with $m \neq n$. Show that, when considered to be functions in $L^2(0,1)$ (i.e., using the $L^2(0,1)$ inner product), these functions are orthogonal, regardless of the choices of m and n. [Hint: Use your result from part (a).] Are two such sine functions still orthogonal when considered as functions in $L^2(-1,1)$? How about $L^2(0,1/2)$?

(c) Compute the value of $\|sin(n\pi\cdot)\|_2$ when $\sin(n\pi\cdot)$ is considered to be a function in $L^2(0,1)$. Redo the calculation now considering it to be a function in $L^2(-1,1)$. Hint: A helpful trigonometric identity here is

$$\sin^2\phi \;=\; \frac{1}{2}\left[1-\cos(2\phi)\right]\ .$$

$\star$11 (a) Below are three pairs of vectors in $\mathbb{R}^2$, given using Octave notation.

$$\mathbf{u}_1 = [3;4],\quad \mathbf{u}_2 = [-4;3]$$
$$\mathbf{u}_1 = [1;0],\quad \mathbf{u}_2 = [0;-2]$$
$$\mathbf{u}_1 = [1;0],\quad \mathbf{u}_2 = [1;1]$$

For each different pair of vectors, do the following

(i) Determine the lengths (norms) of $\mathbf{u}_1$, $\mathbf{u}_2$.

(ii) Write, if possible, the vector $\mathbf{v} = [-8;5]$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$—that is, find scalars c_1, c_2 such that

$$\mathbf{v} \;=\; c_1\mathbf{u}_1 + c_2\mathbf{u}_2\ .$$

(iii) Determine the inner products of $\mathbf{v}$ (given above) with $\mathbf{u}_1$, $\mathbf{u}_2$.

(iv) Indicate whether your answers to parts (i) and (iii) might aid in answering part (ii). Why is the answer sometimes *yes* and sometimes *no*?

(b) In Calculus we learn about the vector projection of one vector $\mathbf{v}$ onto another vector $\mathbf{u}$, given by

$$\mathrm{proj}_{\mathbf{u}}\mathbf{v} \;:=\; \frac{\langle \mathbf{v}, \mathbf{u}\rangle}{\|\mathbf{u}\|^2}\,\mathbf{u}\ .$$

How is what you were doing in part (a) related to this concept?

(c) Projections are fundamental in the function space $L^2(a,b)$ as well:

$$\mathrm{proj}_f g \ := \ \frac{\langle g, f\rangle}{||f||^2} f \ .$$

Suppose we are working in $L^2(0,1)$ and $f(\cdot) = \sin(n\pi\cdot)$ for some integer $n \geq 1$. Show that, no matter the choice of n,

$$\frac{\langle g, f\rangle}{||f||^2} \ = \ 2\int_0^1 g(x)\sin(n\pi x)\,dx \ .$$

3.1.2 (b) Find all separable eigensolutions to the heat equation $u_t = u_{xx}$ on the interval $0 \leq x \leq \pi$ subject to mixed boundary conditions $u(t,0) = 0$, $u_x(t,\pi) = 0$.