

MATH 333: Partial Differential Equations

Problem Set 3, Final version

Due Date: Mon., Sept. 28, 2009

Read Sections 2.4, and 10.1–10.3 from the Olver text. Read Sections 5–7 of the OCTAVE tutorial found at <http://www-mdp.eng.cam.ac.uk/web/CD/engapps/octave/octavetut.pdf>. Please try out the plotting examples, as much has gone into changing the graphics handling in OCTAVE in the last 1–2 years, and some of this is not up-to-date.

2.2.12 A sensor at position $x = 1$ monitors the concentration of a pollutant $u(t, 1)$ as a function of t for $t \geq 0$. Assuming the pollutant is transported with wave speed $c = 3$, at what positions x can you determine the initial concentration $u(0, x)$?

2.2.14 Let $c > 0$. Consider the uniform transport equation $u_t + cu_x = 0$ on the quarter plane $x > 0, t > 0$, subject to initial conditions $u(0, x) = f(x)$ for $x \geq 0$ and boundary conditions $u(t, 0) = g(t)$ for $t \geq 0$.

- For which initial and boundary conditions does a classical solution to this initial-boundary value problem exist? Write down a formula for the solution.
- On which regions are the effects of the initial conditions (boundary conditions) felt? Is there any interaction between the two?

★7 Consider the nonlinear transport problem

$$u_t - 2u_x + u^2 = e^{2x+4t}, \quad 0 < x < 1, \quad 0 < t < 1,$$

subject to the conditions

$$u(0, x) = e^x, \quad u(t, 1) = e^{2t+1}.$$

- Identify the boundary condition, and explain why it makes sense to provide such a condition at the right boundary point instead of the left one. Use your explanation to argue that the backward finite difference approximation for u_x

$$u_x(t, x) \approx \frac{u(t, x) - u(t, x - \Delta x)}{\Delta x}$$

should be considered a “downwind approximation” instead of an “upwind approximation”.

- Verify that the exact solution of the problem is $u(t, x) = e^{x+2t}$.

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- (c) Write a program (probably mimicking, to some extent, the **upwind-scheme program I used in classroom demonstrations**) that solves for the approximate solution via an *upwind* finite difference scheme for a fixed choice of Δt and Δx .
- (d) Using various choices of $\Delta t, \Delta x$, demonstrate that the condition $\Delta t < \Delta x/2$ must be satisfied in order to get a *stable* solution. Are some stable choices of the two able to reproduce the exact solution better than others?
- (e) Solve (again) this problem numerically, but now employing a *downwind* finite difference scheme. Are there any choices of $\Delta t, \Delta x$ which lead to an acceptable solution?