MATH 333: Partial Differential Equations

Problem Set 2, Final version

Due Date: Mon., Sept. 21, 2009

Read Sections 2.1–2.2 from the Olver text.

1.13 Suppose $u(t,x)$, $v(t,x)$ are C^2 functions defined on $\mathbb{R}^2$ and satisfy the first order system of partial differential equations $u_t = v_x$, $v_t = u_x$.

(a) Show that both u and v are classical solutions to the wave equation $w_{tt} = w_{xx}$. Which result from vector calculus do you need to justify the conclusion?

(b) Conversely, given a classical solution $u(t,x)$ to the wave equation, can you construct a function $v(t,x)$ such that $u(t,x)$, $v(t,x)$ form a solution to the first order system?

1.16 Classify the following differential equations as either (i) homogeneous linear; (ii) inhomogeneous linear; or (iii) nonlinear.

(a) $u_t = x^2 u_{xx} + 2x u_x$,

(b) $-u_{xx} - u_{yy} = \sin u$,

(c) $u_{xx} + 2y u_{yy} = 3$,

(d) $u_t + u u_x = 3u$,

(e) $e^y u_x = e^x u_y$,

(f) $u_t = 5u_{xxx} + x^2 u + x$.

1.19 The displacement $u(t,x)$ of a forced violin string is modeled by the partial differential equation $u_{tt} = 4u_{xx} + F(t,x)$. When subjected to the external forcing $F(t,x) = \cos x$, the solution is $u(t,x) = \cos(x - 2t) + \frac{1}{4}\cos x$, while when $F(t,x) = \sin x$, the solution is $u(t,x) = \sin(x - 2t) + \frac{1}{4}\sin x$. Find a solution when the forcing function $F(t,x)$ is

(a) $\cos x - 5\sin x$,

(b) $\sin(x - 3)$.

1.20 (a) Show that the partial derivatives $\partial_x[f] := \dfrac{\partial f}{\partial x}$ and $\partial_y[f] := \dfrac{\partial f}{\partial y}$ both define linear operators on the space of continuously differentiable functions $f(x,y)$.

(b) For which values of a, b, c, d is the map $L[f] := a\dfrac{\partial f}{\partial x} + b\dfrac{\partial f}{\partial y} + cf + d$ linear?

2.1.3 Find the general solution $u(t,x)$ to the following PDEs. Consider the domain of definition to be $x \in \mathbb{R}$, $t > 0$.

(c) $u_t = x - t$,

(e) $u_x + tu = 0$,

(f) $u_{tt} + 4u = 1$.

2.1.6 Solve the partial differential equation $\dfrac{\partial^2 u}{\partial x\, \partial y} = 0$, $(x, y) \in \mathbb{R}^2$, for $u(x, y)$.

2.2.2 Solve the initial value problems and graph the solutions at times $t = 1, 2$ and 3:

(a) $u_t - 3u_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(0, x) = e^{-x^2}$;

(c) $u_t + u_x + \dfrac{1}{2} u = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(0, x) = \arctan x$.

2.2.5 Solve the initial value problem $u_t + 2u_x = \sin x, \quad x \in \mathbb{R}, \quad t > 0, \quad u(0, x) = \sin x$.

2.2.17 (a) Solve the initial value problem $u_t - xu_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \quad u(0, x) = (x^2 + 1)^{-1}$.

(b) Graph the solution at time $t = 0, 1, 2, 3$.

(c) What is $\lim_{t \to 0} u(t, x)$?