

MATH 333: Partial Differential Equations

Project 2, Due Date: Fri., Sept. 29, 2006

Consider the vibrations of an infinitely long thin string (one extending from position $x = 0$ up to $x = +\infty$). The PDE model for this problem is

$$\begin{aligned}u_{tt} &= c^2 u_{xx}, & x > 0, \quad t > 0, \\u(0, t) &= 0, & t > 0, \\u(x, 0) &= \phi(x), & x > 0, \\u_t(x, 0) &= \psi(x), & x > 0.\end{aligned}$$

The solution of this problem is given by

$$u(x, t) = \begin{cases} \frac{1}{2} [\phi(x + ct) - \phi(ct - x)] + \frac{1}{2c} \int_{ct-x}^{x+ct} \psi(s) ds, & \text{if } 0 < x < ct, \\ \frac{1}{2} [\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds, & \text{if } x > ct. \end{cases} \quad (1)$$

1. For the specific instance in which, $c = 0.5$,

$$\phi(x) = xe^{-x} \quad \text{and} \quad \psi(x) \equiv 0,$$

write out a more explicit formula for this solution. Then plot the solution on the interval $0 \leq x \leq 10$ for various values of t between 0 and 20 to get a feel for the nature of the traveling wave solution. (It is best if you can make a movie out of it, but I might recommend MATHEMATICA for that.) Does the nature of the solution fit well with your experience, say, of a cord that is tied to a door handle?

2. Find a resource which explains the derivation of solution (1) (a key phrase to look for might be the “wave equation on a semi-infinite domain), write out this derivation, and expound on the details so that you have notes on this for future reference.