MATH 333: Partial Differential Equations

Project 10, Due Date: Mon., Nov. 17, 2006

(a) Under the usual inner product of $L^2(a,b)$, the family $S = \{x^n \,|\, n = 0, 1, 2, \ldots\}$ of monomials is not mutually orthogonal. Show that this is so in the particular case where $a = -1$ and $b = 1$.

(b) Suppose one is given a set of vectors $\{u_1, u_2, \ldots\}$ from an inner product space V. There is a process known as Gram-Schmidt orthogonalization which uses such a *starter* set to build a (new) set $\{v_1, v_2, \ldots\}$ of orthogonal vectors in V. One starts by simply taking $v_1 := u_1$. Then, one iteratively takes

$$v_{n+1} \;:=\; u_{n+1} - \sum_{k=1}^{n} \frac{\langle u_{n+1}, v_k \rangle}{\|v_k\|^2}\, v_k.$$

(If/when any of the v_k's formed this way are the zero vector/function, they are thrown out.) Explain why the resulting v_ks must be mutually orthogonal. Then carry out the process on the set $\{1, x, x^2, x^3, x^4\}$ from the previous part, considering these monomials to be functions in $V = L^2(-1,1)$. Note that, if v_j and v_k are orthogonal, then so are cv_j and dv_k for any nonzero scalars c and d. Multiply each of the functions that result from your Gram-Schmidt process on the monomials above by an appropriate constant in order that each has a value of 1 at $x = 1$. You have found the nth-degree Legendre polynomials $P_n(x)$ for $n = 0, 1, 2, 3$, and 4.

(c) Find the polynomial of degree at most 4 that is "closest" in the $L^2(-1,1)$-norm among all 4th-degree polynomials to the function $1/(1+x^2)$. How does your answer compare with the 4th-degree Taylor polynomial of $1/(1+x^2)$ at $x = 0$?

(d) Consider the family of Sturm-Liouville DEs (one for each choice of $n = 0, 1, 2, \ldots$)

$$(1-x^2)y'' - xy' + n^2 y \;=\; 0.$$

(Note that the coefficient functions almost meet the requirements for regularity.) Assume that, for n fixed, the DE has a solution in the form of an nth-degree polynomial

$$T_n(x) \;=\; a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \qquad \text{with} \qquad a_n \;=\; 2^{n-1}.$$

Use these facts to find the polynomials T_0, T_1, T_2, T_3, and T_4, known as Tchebyshev polynomials.

(e) Though not coming from a regular Sturm-Liouville problem, show that $T_0, \ldots T_4$ are mutually orthogonal in the space $L^2_w(-1,1)$ made up of of functions f defined almost everywhere on $(-1,1)$ having finite weighted L^2-norm given by

$$\|f\|_{2,w} \;:=\; \int_{-1}^{1} |f(x)|^2 w(x)\, dx, \qquad \text{with} \qquad w(x) \;:=\; \frac{1}{\sqrt{1-x^2}}.$$

(Note: First determine what is the inner product that gives rise to this norm.) Does there seem to be a common value for $\|T_n(x)\|_{2,w}^2$ independent of n, or does this norm change with n?

(f) The Tchebyshev polynomials have the alternate expression $T_n(x) = \cos(n\cos^{-1}(x))$. Demonstrate first that this is plausible by showing that this expression satisfies the DE in part (d).

(g) When $n \geq 3$, it is rare for an nth-degree polynomial chosen at random to have n real zeros. Using the alternate expression from part (f), show that T_n

  (i) has n distinct real zeros, all of which lie in $[-1, 1]$.

  (ii) has $n-1$ distinct relative maxima in $[-1, 1]$. Find the sup-norm of T_n over $[-1, 1]$.

  (iii) satisfies the 2nd-order recursion relation:

$$T_0(x) \;=\; 1, \quad T_1(x) \;=\; x, \quad \text{and} \quad T_{n+1}(x) \;=\; 2xT_n(x) - T_{n-1}(x), \;\; n \geq 1.$$

  Hint: Use the trigonometric identities

$$\cos((n+1)\theta) \;=\; \cos\theta\cos(n\theta) - \sin\theta\sin(n\theta),$$

  and

$$\cos((n-1)\theta) \;=\; \cos\theta\cos(n\theta) + \sin\theta\sin(n\theta).$$

  It should be clear from this recursion relation that the expressions in part (f) are, indeed, nth-degree polynomials with leading coefficient 2^{n-1}. Since they satisfy the DE in (d), they must be the same functions as you were finding in (d).

(h) For an extra amount (0.5-1 point), show that, among all monic polynomials (ones whose leading coefficient is 1) of degree n, $T_n(x)/2^{n-1}$ is the one possessing the smallest ∞-norm in $[-1, 1]$.