MATH 333: Partial Differential Equations

Project 9, Due Date: Mon., Nov. 13, 2006

Because we have placed greater emphasis than in past semesters upon the numerical solution of PDEs, we have correspondingly placed less emphasis upon the eigenfunctions that arise from different types of domains and BCs. This project gives you the opportunity to explore this on your own. The different parts can all be done standalone—that is, you can earn the number of points indicated on the particular parts you do without doing all parts.

1. (*0.5 point*) Consider the diffusion problem

$$\begin{aligned} u_t &= k u_{xx}, && 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, && t > 0, \\ u(1,t) + u_x(1,t) &= 0, && t > 0, \\ u(x,0) &= f(x), && 0 < x < 1. \end{aligned}$$

   The boundary condition at the right endpoint is called a Robin condition. Use separation of variables to find a formal solution of the problem. You will not be able to find exactly the eigenvalues of the corresponding SLEP, but use SLEP theory to justify the steps you make wherever appropriate. Determine the first n eigenvalues numerically, and then plot at various times the truncated Fourier series solutions (i.e., solutions involving sums of terms up to N rather than ∞) for various choices of N in the case $f(x) = 10x(1-x)$.

2. (*0.5 point*) Suppose a solid ball of radius a is dropped into a bath held at a fixed temperature. We wish to solve for the temperature of the ball as it changes in time. While such a problem seems to rely on 4 independent variables (time and 3 spatial dimensions), if we make the simplifying assumption that the initial temperature profile is radially symmetric, then it is reasonable to assume the solution *remains* radially symmetric, relying only on time and distance ρ from the ball's center. This leads us to the model problem

$$\begin{aligned} u_t &= \Delta u, && 0 \le \rho < a, \quad t > 0, \\ u(a,t) &= 0, && t > 0, \\ u(\rho,0) &= f(\rho), && t > 0. \end{aligned}$$

   Write out the Laplacian in spherical coordinates, taking advantage of the assumed symmetries in the problem. Then make the substitution $v(\rho,t) = \rho u(\rho,t)$ and first solve the problem for v. Sketch the solution u at various times when $f(\rho) = 1$.

3. (*1 point*) Consider the transverse vibrations of the Bernoulli-Euler beam. For small deflections, the Bernoulli-Euler model

$$m(x)\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] = f(x,t),$$

fairly accurately predicts the vertical deflection $u(x,t)$ from rest of the horizontal beam; here $m(x)$ is the mass density per unit length, $EI(x)$ is the flexural rigidity density, and $f(x,t)$ is an external deflecting force density. There are several common boundary conditions used with this model, here written as if they apply to the left endpoint $x=0$:

(i) *clamped (fixed)*: $u(0,t) \;=\; u_x(0,t) \;=\; 0,$

(ii) *pinned (hinged)*: $u(0,t) \;=\; u_{xx}(0,t) \;=\; 0,$ or

(iii) *free (floating)*: $u_{xx}(0,t) \;=\; u_{xxx}(0,t) \;=\; 0.$

Assume the simplest (nondimensional) model for the transverse vibrations of a freely vibrating cantilevered homogeneous beam

$$\frac{\partial^2 u}{\partial t^2} \;=\; -\frac{\partial^4 u}{\partial x^4}, \quad 0 < x < 1, \;\; t > 0,$$

subject to the clamped conditions at the left endpoint, the free conditions at the right end ($x = 1$), initial displacement and velocities $u(x,0) = f(x)$ and $u_t(x,0) = 0$ respectively. Assume separation of variables and find the corresponding spatial BVP. It is possible (and this is part of your task) to show that the eigenvalues of the operator d^4/dx^4 are real and nonnegative, and writing them as $\lambda = \beta^4$ for real β, that both eigenvalues and eigenfunctions may be characterized by the solutions β_k of an equation in β. (Note that, as $k \to \infty$, the correct $\beta_k \to (2k-1)\pi/2$.) Write a program in OCTAVE that takes the first n of these zeros and produces a truncated Fourier series solution (so it finds the approximate Fourier coefficients), and graphs that approximate solution at any desired time t. Assume that the initial displacement is $f(x) = (x^4 - 4x^3 + 6x^2)/20$. Can you make a movie in OCTAVE that demonstrates the vibrations?

4. (*up to 2 points*) Consider the vibrations of a circular membrane (drum), modeled by the equation

$$\frac{\partial^2 u}{\partial t^2} \;=\; c^2 \Delta u.$$

The geometry suggests polar coordinates, or that $u = u(r,\theta,t)$. We assume the membrane is fixed at the circular boundary, and the initial position and velocity are given:

$$u(a,\theta,t) \;=\; 0, \qquad \text{and} \qquad \begin{array}{rcl} u(r,\theta,0) &=& \alpha(r,\theta), \\ u_t(r,\theta,0) &=& \beta(r,\theta). \end{array} .$$

First, assume separation $u(r,\theta,t) = \phi(r,\theta)T(t)$ (with separation constant $(-\lambda)$), and find corresponding DEs in t (ordinary) and (r,θ) (partial). What BC goes with the latter? Can you show that $\lambda > 0$ holds? Then assume further separation $\phi(r,\theta) = f(r)g(\theta)$ (another separation constant, μ), writing the Laplacian operator in polar form. The resulting ODE in θ should have periodic boundary conditions:

$$g(-\pi) \;=\; g(\pi) \qquad \text{and} \qquad g'(-\pi) = g'(\pi).$$

It seems we are lacking a BC for the ODE in r, but the extra condition will be that $f(r)$ remains bounded as $r \to 0$. Solve the θ BVP first to get eigenvalues μ_m. Next, turn to the BVP in r, which contains both separation constants (now with $\mu = \mu_m$). Show that this problem can be written in the Sturm-Liouville form

$$\frac{d}{dr}\left(r\,\frac{df}{dr}\right) + \left(\lambda r - \frac{m^2}{r}\right) f \;=\; 0.$$

Though this is not a regular SLEP, it is a fact that our statements about the eigenvalues/eigenfunctions of regular SLEPs still hold. Make the substitution $z = r\sqrt{\lambda}$ to arrive at Bessel's differential equation of order m, which has general solution

$$f \;=\; c_1 J_m(z) + c_2 Y_m(z),$$

where J_m, Y_m are called the Bessel functions of the first, second kind of order m respectively. Note that $\lim_{z\to 0} Y_m(z) = \infty$. The graphs of J_m look like decaying oscillations with infinitely many zeros. You can see such a graph for, say, J_0 in OCTAVE by typing

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> x = [0:.1:20]';
> plot(x, besselj(0,x))
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How are the eigenvalues λ related to these zeros? (Note that they should be doubly-indexed: $\lambda = \lambda_{mn}$.) What are the corresponding eigenfunctions? Stepping back up to the 2-dimensional level, what are the eigenfunctions $\phi_\lambda = \phi_{mn}$? Compare these eigenfunctions to those of a vibrating square membrane. (First, you must state the precise square membrane problem, then solve it via separation of variables.)

A similar analysis may be carried out in the case where vibrations are circularly symmetric—that is,

$$\begin{aligned} u_{tt} &= c^2 r^{-1}\left(r u_r\right)_r, \quad 0 \le r < a, \;\; t > 0, \\ u(a,t) &= 0, \quad t > 0, \\ u(r,0) &= \alpha(r), \;\; u_t(r,0) \;=\; \beta(r), \quad 0 \le r < a. \end{aligned}$$

In this case there is no need for a 2nd separation constant. Deduce that

$$u(r,t) \;=\; \sum_{k=1}^{\infty} a_k J_0(\sqrt{\lambda_k} r)\cos(c\sqrt{\lambda_k} t) + \sum_{k=1}^{\infty} b_k J_0(\sqrt{\lambda_k} r)\sin(c\sqrt{\lambda_k} t),$$

where

$$a_k \;=\; \frac{\displaystyle\int_0^a \alpha(r) J_0(\sqrt{\lambda_k} r) r\,dr}{\displaystyle\int_0^a J_0^2(\sqrt{\lambda_k} r) r\,dr}.$$

Derive a formula as well for the b_k in terms of $\beta(\cdot)$. Find a resource online (perhaps http://webcomputing.bio.bas.bg/webMathematica/webComputing/BesselZeros.jsp) for the zeros of Bessel functions of the first kind. Let the radius of the circle $a = 1$, and use the zeros of J_0 to find the first 20 coefficients a_k in the expansion

$$\sin(1 - r^2) \ = \ \sum_{k=1}^{\infty} a_k J_0(\sqrt{\lambda_k} r).$$

Then sketch several snapshots in time to the solution of the circularly symmetric vibrating membrane problem with $\alpha(r) = \sin(1 - r^2)$ and $\beta(r) = 0$. Can you animate these vibrations?