MATH 333: Partial Differential Equations

Problem Set 13, Final version

Due Date: Wed., Dec. 6, 2006

1. Chapter 7 of our book is devoted to the Dirichlet problem for Poisson's equation in two spatial dimensions
$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega \subset \mathbb{R}^2, \\ u &= g, \quad \text{in } \partial\Omega. \end{aligned}$$
Section 7.5 is the portion of the chapter that deals with numerical solutions of this problem using finite differences on a grid Ω_h. For Example 7.2, p. 228, which is the one to which we applied a Galerkin approach in class, the specifics are that $\Omega = \{(x,y) \mid 0 < x < 1,\ \ 0 < y < 1\}$, $g(x,y) \equiv 0$, and
$$f(x,y) = [(3x + x^2)y(1-y) + (3y + y^2)x(1-x)]e^{x+y}.$$
The grid Ω_h for this problem is pictured on p. 226, and the true solution is known to be
$$u(x,y) = x(1-x)y(1-y)e^{x+y}.$$
   (a) Table 7.1, appearing at the top of p. 229, contains information comparing the approximate solution of the above problem via finite differences to the true solution. While it is not my goal that you learn the finite difference approach of Section 7.5, scan this section as needed, and then write a couple sentences about how one should interpret the various values for E_h provided there.
   (b) Using just one trial function in the Galerkin approach yields the approximate solution
$$\tilde{u}_1(x,y) = -360(57 - 40e + 7e^2)(x - x^2)(y - y^2).$$
Numerically estimate the value of
$$\sup_{(x,y)\in\overline{\Omega}} |u(x,y) - \tilde{u}_1(x,y)|.$$
Here $\overline{\Omega} = \Omega \cup \partial\Omega$ (called the *closure* of Ω), and u denotes the true solution (given above) of the problem. How does this error estimate compare with the information from Table 7.1? Try to do produce a similar error estimate when four trial functions are used, yielding approximate solution
$$\begin{aligned} \tilde{u}_2(x,y) &= c_{11}(x - x^2)(y - y^2) + c_{12}(x - x^2)(y^2 - y^3) \\ &\quad + c_{21}(x^2 - x^3)(y - y^2) + c_{22}(x^2 - x^3)(y^2 - y^3), \end{aligned}$$
with
$$\begin{aligned} c_{11} &= -\frac{1440}{13}(431849 - 317790e + 58464e^2), \\ c_{12} &= c_{21} = \frac{50400}{13}(24251 - 17851e + 3285e^2), \\ c_{22} &= -50400(3667 - 2700e + 497e^2). \end{aligned}$$

As a reminder from MATH 162, recall that real-valued differentiable functions of multiple variables $f(x_1, \ldots, x_n)$ have local extrema only at *critical points*, or points at which $\nabla f = \mathbf{0}$ (i.e., where all components $\partial f/\partial x_1$, …, $\partial f/\partial x_n$ of the gradient vector are simultaneously zero). For the function $u - \tilde{u}_1$, it is easy enough to *find* the partial derivatives with respect to x and y, but another matter to solve (either by hand, or in *Mathematica*) the resulting equations for critical points. The most accurate estimations of these critical points will come from a root-finding algorithm, such as Newton's method. Should you want to use *Mathematica* to try this, here is a notebook I've used in MATH 162 to demonstrate the locating of critical points for functions of two variables. I'm afraid the **Solve**[ ] command (found therein) will fail miserably when applied to the function whose extrema you want, but try **FindRoot**[ ] in its place. (On the *Mathematica* help for **FindRoot**, the most relevant example is the last one shown.) I suggest looking at graphs of $u - u_1$ and $u - u_2$ (see **Plot3D**[ ]) to determine initial guesses for the root finder. I believe it is just as straightforward numerically solving for the critical points in OCTAVE using the `fsolve` command, particularly if you're comfortable in that framework producing plots of functions of two variables (see my solution to Problem 3 in Problem Set 10 for an example) in order to find initial guesses.

If you wish to play around with the *Mathematica* notebook I used in class to implement a Galerkin solution to this problem, here is a link to it.

2. In class, we started Example 5 from the handout (Chapter 13) entitled "Galerkin's Method". We have seen that the coefficients c_j for the Galerkin solution (13.38) are found by solving a matrix equation $A\boldsymbol{c} = \boldsymbol{b}$, where the (i,j)th component of A is $\langle \phi_i, L\phi_j \rangle$, and $L\phi = \phi'' + \phi$. Nevertheless, for our "hat" functions ϕ_j, the second derivative is zero almost everywhere (and, in fact, the 2nd derivative is not an L^2 function). The remedy is to rewrite these inner products using integration by parts:

$$\begin{aligned}
\langle \phi_i, Ly \rangle &= \int_0^1 \phi_i(x)[y''(x) + y(x)]\,dx \\
&= \int_0^1 y''(x)\phi_i(x)\,dx + \int_0^1 y(x)\phi_i(x)\,dx \\
&= y'(x)\phi_i(x)\Big|_0^1 - \int_0^1 y'(x)\phi_i'(x)\,dx + \int_0^1 y(x)\phi_i(x)\,dx \\
&= \int_0^1 y(x)\phi_i(x)\,dx - \int_0^1 y'(x)\phi_i'(x)\,dx \\
&= \langle \phi_i, y \rangle - \langle \phi_i', y' \rangle\,,
\end{aligned}$$

where the "boundary terms" $y'(x)\phi_i(x)\Big|_0^1$ drop out because we require the test functions to satisfy the homogeneous BCs. Thus, the strict requirement that the trial functions be in the domain of the operator L (i.e, be twice differentiable) is replaced with the weaker (and more favorable, given the trial functions we are using) requirement that they be differentiable. Now the (i,j)th entry of A is

$$\langle \phi_i, \phi_j \rangle - \left\langle \phi_i', \phi_j' \right\rangle,$$

and we may split A into the difference of matrices $G-S$ (G is called a *Gram* matrix, S a *Sobolev* matrix), where the (i,j)th entries of G, S are $\langle \phi_i, \phi_j \rangle$ and $\left\langle \phi_i', \phi_j' \right\rangle$ respectively.

(a) Do Exercise 13.12, which is to verify the expressions for the entries in G, S found in equations (13.44) and (13.45).

(b) Building on "Routine 13.3" (middle of p. 255), write OCTAVE code to produce graphs of Galerkin finite-element solutions to

$$y'' + y \;=\; -1, \;\; 0 < x < 1, \quad \text{subject to} \quad y(0) = y(1) = 0,$$

like those displayed on p. 256 of the handout.

3. (a) Suppose f, g are real-valued, differentiable functions on an interval $[a,b]$. Use the product rule and the Fundamental Theorem of Calculus, Part II to derive/prove the integration by parts formula (Equation (1) on the "Some Notes on Vector Calculus" handout) for definite integrals.

   (b) Use the Divergence theorem and the product rule for dot products to derive/prove Green's first identity.

F1: *This is a final exam problem. Bring your answer with you to the final. Do not discuss the problem at any stage with other members of the class.*
In Problem 1(b) of Problem Set 10, we found the closest 4th-degree polynomial to $1/(1+x^2)$, as measured by the $L^2(-1,1)$-norm of the difference. In this problem, we undertake a similar task (albeit in a different manner), this time seeking to minimize the $L^2(0,1)$ norm of the difference. Let $f \in \mathcal{C}([0,1])$ be the function we intend to approximate, and let

$$\tilde{f}(x) \;=\; \sum_{k=0}^{n} a_k x^k.$$

If we define

$$E(a_0, a_1, \ldots, a_n) \;:=\; \int_0^1 \left[f(x) - \sum_{k=0}^{n} a_k x^k \right]^2 dx$$

then by choosing scalars $a_0, \ldots, a_n$ that minimize E, we will be minimizing the $L^2(0,1)$ norm of the difference $(f - \tilde{f})$. From calculus we know that a necessary condition for minimizing E over all choices $(a_0, a_1, \ldots, a_n) \in \mathbb{R}^{n+1}$ is that

$$\frac{\partial E}{\partial a_j} \;=\; 0, \quad \text{for } \; j = 0, \ldots, n.$$

These $(n+1)$ equations are called the *normal equations* for this "least squares problem".

(a) Show that the normal equations (in the $(n+1)$ unknowns $a_0, \ldots, a_n$) can be written as

$$\sum_{k=0}^{n} a_k \left\langle x^k, x^j \right\rangle = \left\langle f, x^j \right\rangle, \quad j = 0, 1, \ldots, n,$$

where $\langle \cdot, \cdot \rangle$ denotes the $L^2(0,1)$ inner product.

(b) Show that the normal equations may be recast as a matrix problem $H\boldsymbol{a} = \boldsymbol{b}$, where $\boldsymbol{a}$ is a $(n+1)$-by-1 vector whose jth component is a_{j+1}, $\boldsymbol{b}$ is also $(n+1)$-by-1, and the entry in the (i,j)th position of the matrix H is

$$h_{ij} = \frac{1}{i+j-1}.$$

(c) In our discussion of the Galerkin method, we have been solving a differential equation $Lu = f$ where L is a linear operator. Galerkin has led to a matrix problem much like the one above. What operator L would make the matrix equation above precisely that of the Galerkin method applied to $Lu = f$ using test functions x^j, $j = 0, 1, \ldots, n$?

(d) The matrix H of part (b) is a common matrix called the $(n+1)$-square *Hilbert matrix*. Octave comes equipped with the command `hilb()`, which produces H to whatever (square) dimension you desire.

Once again, let's approximate $f(x) = (1+x^2)^{-1}$ with nth-degree polynomials. Write a routine in Octave that accepts the single argument n, and returns the coefficients $a_0, \ldots, a_n$ which minimize E. Then use the `polyval()` function to produce plots of the resulting polynomial for various choices of n, each time plotting it together with f. The syntax for `polyval` is that it accepts two inputs, both of which can be vectors:

> `polyval([1 3 2]', -1)` returns the value of the polynomial $1+3x+2x^2$ at $x = -1$;
> `polyval([1 0 -1 1], [0:.1:1])` returns a vector of values for $1 - x^2 + x^3$ corresponding to the inputs `[0 0.1 ... 1]`.

Using the values provided in the following table for $\langle f, x^j \rangle$, $j = 0, \ldots, 11$, determine the first choice of n for which the polynomial approximation stops having any validity. See this wiki entry for a (very) brief explanation as to why this loss of usefulness occurs.

j	0	1	2	3
$\langle f, x^j \rangle$	$\pi/4$	$\ln(2)/2$	$1 - \pi/4$	$(1 - \ln 2)/2$
j	4	5	6	7
$\langle f, x^j \rangle$	$\pi/4 - 2/3$	$[\ln(4) - 1]/4$	$13/15 - \pi/4$	$(5 - 6\ln 2)/12$
j	8	9	10	11
$\langle f, x^j \rangle$	$\pi/4 - 76/105$	$[12\ln(2) - 7]/24$	$263/315 - \pi/4$	$(47 - 60\ln 2)/120$