MATH 333: Partial Differential Equations

Problem Set 11, Final version

Due Date: Wed., Nov. 22, 2006

1. We have, up until now, been dealing almost exclusively with what are called *real-valued functions*, functions whose outputs are real numbers (usually their inputs are as well). One notable exception is the complex-valued function

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

While its inputs θ are presumed real, the outputs are real only for special choices of θ ($\theta = n\pi$, $n = \ldots, -1, 0, 1, \ldots$). One often writes a complex-valued function in terms of real-valued component functions, as in

$$f(x) = u(x) + iv(x), \tag{1}$$

which says that the *real part* of the output $f(x)$ is found by evaluating $u(x)$, while the *imaginary part* is found evaluating $v(x)$. One can summarize relationship (1) by writing $\operatorname{Re} f = u$ and $\operatorname{Im} f = v$. Insisting only that differentiation and integration continue to be linear operations when applied to complex-valued functions leads to the conclusion that

$$\frac{d}{dx}f(x) = \frac{d}{dx}u(x) + i\frac{d}{dx}v(x) = u'(x) + iv'(x),$$

and

$$\int_a^b f(x)\,dx = \int_a^b u(x)\,dx + i\int_a^b v(x)\,dx.$$

People tend to plot real numbers on a number line; when asked about the magnitude of a real number x, usually the answer $|x|$ is given, representing the number's distance from 0. Similarly, people tend to plot complex numbers $x + iy$ on a plane, in exactly the same location as the coordinate pair (x, y); the magnitude of $x + iy$, still denoted $|x + iy|$, is still taken to be the distance, given by $\sqrt{x^2 + y^2}$, from that point to the origin. If $z = x + iy$, a useful relationship is $|z| = \sqrt{z\bar{z}}$, where $\bar{z} = x - iy$ is called the *complex conjugate* of z. The proof of this relationship:

$$|z| = |x + iy| = \sqrt{x^2 + y^2} = \sqrt{(x + iy)(x - iy)} = \sqrt{z\bar{z}}.$$

(a) Show that, for a complex-valued f, the inner product of $L^2(-1, 1)$ we are accustomed to

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

does not give rise to a norm $\|f\| = \sqrt{\langle f, f\rangle}$.

(b) Show that the new definition

$$\langle f, g \rangle \;:=\; \int_{-1}^{1} f(x)\overline{g(x)}\,dx$$

does give rise to a nonnegative-valued function $\|\cdot\|$ which can only be zero when the input function is zero.

(c) Find in your notes (first week of class) the properties we listed for inner products. Check the newly-defined operation $\langle\cdot,\cdot\rangle$ (from the previous part) against these properties and find the one that is not satisfied. Can you find a way to state this property anew so that it is unchanged from the original when the arguments to $\langle\cdot,\cdot\rangle$ are real-valued functions, but so that your newly-stated property is also satisfied when complex-valued functions are used as arguments?

(d) Show that, under our new inner product on $L^2(-1,1)$, the collection $S = \{e^{ik\pi x} \mid k = \ldots,-1,0,1,2\ldots\}$ has mutually orthogonal elements, all having the same length (norm).

(e) The collection S from the previous part is a complete orthogonal basis on the space $L^2(-1,1)$ (which, though there was no point in mentioning it before, includes not only real-valued functions, but also complex-valued ones defined almost everywhere on $(-1,1)$ and having finite L^2-norm). So, each $f \in L^2(-1,1)$ may be expressed as

$$f(x) \;=\; \sum_{k=-\infty}^{\infty} c_k e^{ik\pi x},$$

with convergence in (at least) the mean square sense. Use the results of the previous part to find expressions for the coefficients c_k.

(f) We know that each $f \in L^2(-1,1)$ may be expressed in terms of the complete orthogonal basis $S' = \{1, \sin(\pi x), \cos(\pi x), \sin(2\pi x), \cos(2\pi x), \ldots\}$, via

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}[a_k\cos(k\pi x) + b_k\sin(k\pi x)],$$

with the a_k's, b_k's given by formulas you derived in Exercise 3.15. What is the relationship between the a_k's, b_k's from this series expansion and the c_k's from the previous one? (Answer this both by showing how one can use the a_k's and b_k's to get the c_k's, and how one may use the c_k's to determine the a_k's and b_k's.) Why should this relationship hold? In general, the c_k's can be (non-real) complex numbers even when f is real-valued. Are there special types of real-valued f whose coefficients c_k are all real? What are they?

2. Do either Exercise 4.6 or Exercise 4.7 (you choose; the implicit scheme of 4.7 is likely to be more useful). For approximating the Neumann BCs, I recommend something akin to Scheme S_1 as described in Exercise 2.14 (from Problem Set 5).

3. The Crank-Nicholson Scheme. Read (but you need not do) Exercise 4.15. Then do Exercise 4.16, parts (a)–(c) and (e). For part (c), see the class notes (parts of two different days) which showed our explicit method for the heat problem on $[0, 1]$ subject to homogeneous Dirichlet conditions required $r < 1/2$; these class notes mirror Sections 4.2.1–4.2.3.