

MATH 333: Partial Differential Equations

Problem Set 9, Final version

Due Date: Wed., Nov. 8, 2006

1. If we combine our assertion (made without proof) in class that the collection $S = \{1, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \dots\}$ is a complete orthogonal set on $L^2(-1, 1)$ with the results of Exercise 3.15 (see Problem Set 8), then we get that any function $f \in L^2(-1, 1)$ is equal on $[-1, 1]$ (in the mean-square convergence sense) to its Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\pi x) + b_k \sin(k\pi x)],$$

with

$$a_k = \int_{-1}^1 f(x) \cos(k\pi x) dx \quad \text{and} \quad b_k = \int_{-1}^1 f(x) \sin(k\pi x) dx.$$

More generally, given any function $f \in L^2(-\ell, \ell)$ with $\ell > 0$, then on $[-\ell, \ell]$ we have

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} [a_k \cos(k\pi x/\ell) + b_k \sin(k\pi x/\ell)],$$

(with equality, again, understood in the mean-square convergence sense), with the slightly altered formulas for the Fourier coefficients

$$a_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos(k\pi x/\ell) dx \quad \text{and} \quad b_k = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin(k\pi x/\ell) dx.$$

I have written functions in OCTAVE which can help demonstrate this. The script `fourierCoeffs.m` must be called with three arguments: the function whose Fourier series we seek, the value of ℓ , and the number of Fourier coefficients to calculate. Typing commands in OCTAVE such as

```
octave:1> function y = f(x)
> y = (-x-1.5).*(x<-1.5) + (x+.5).*(x>-0.5 & x<0.5) + ...
> (x>=.5 & x<1.5) + (2.5-x).*(x>=1.5);
> end
```

```
octave:2> [ak_s, bk_s] = fourierCoeffs('f', 2, 5);
```

will return the coefficients a_0, \dots, a_5 stored in a length-6 vector `ak_s` and the coefficients b_1, \dots, b_5 , stored in the length-5 vector `bk_s`. Armed with the Fourier coefficients $a_0, \dots, a_N, b_1, \dots, b_N$, one might wish to plot the *truncated Fourier series* $a_0/2 + \sum_{k=1}^N [a_k \cos(k\pi x/\ell) + b_k \sin(k\pi x/\ell)]$ of f . The code

```
octave:3> x = [-2:.01:2]';
octave:4> plot(x, f(x), 'b', x, fs_approx('f', x, 2, 7), 'r')
```

will plot both the function f (which we had already defined) and the truncated Fourier series (here with $N = 7$) on the same graph for $x \in [-2, 2]$. (Note that `fs_approx.m` is another script which takes care of calling the script `fourierCoeffs.m`, so that no prior call to the latter is required.)

- (a) Download the OCTAVE scripts mentioned above, as well as `f_times_cosine.m`, `f_times_sine.m` and `my_quadg.m`, all found at <http://www.calvin.edu/~scofield/courses/m333/materials/octave/>, placing them in your working directory (the one from which you start OCTAVE). You will see calls to all three of the latter scripts made from within `fourierCoeffs.m`. In particular, `my_quadg.m` does pretty much what the OCTAVE function `quad` does (see Problem Set 3), with the important additional feature that any arguments beyond the first 5 are passed directly along to the function being integrated. The reason this is necessary: the Fourier coefficients come from integrals with integrands like $f(x) \cos(k\pi x/\ell)$ and $f(x) \sin(k\pi x/\ell)$. Such integrands really require two inputs: the x where we wish to evaluate them, and the integer k . Notice that both these inputs are accepted by the script `f_times_sine.m` (along with two additional arguments), and that the call within `fourierCoeffs.m` to `my_quadg.m` includes 3 additional arguments (beyond the 5 that `quad` would have accepted) which are simply passed on to the integrand. (FYI: I believe that `my_quadg.m` will run correctly on all machines, but am not certain. If the OCTAVE command `which quadg` returns a path to a currently-installed script called `quadg.m` rather than an error message, things should be fine.)

Experiment with different calls to `fs_approx.m`, such as the one above. Try changing the number of terms included in the truncated Fourier series to see how increases in the number of terms affects the approximation. (Warning: Don't bump up the number of terms included by too much at once until you get a feel for how long it takes to produce the corresponding plots.) Then run commands like

```
octave:5> x = [-2:.01:2]';
octave:6> plot(x, f(x), 'b', x, fs_approx('f', x, 1, 7), 'r')
```

```
octave:7> x = [-4:.01:4]';
octave:8> plot(x, f(x), 'b', x, fs_approx('f', x, 2, 7), 'r')
```

and the like. Explain the phenomena you witness. In particular, describe what happens when the ℓ included in the call to `fs_approx.m` is smaller than half the width of the interval discretized in the vector x , and explain this behavior. Do you witness any instances of Gibbs' phenomenon (described in Section 9.3 of our text, or at http://en.wikipedia.org/wiki/Gibbs_phenomenon)? Does the phenomenon disappear as you include more terms? Where can it be expected to occur?

- (b) We have asserted that $\{\sin(k\pi x/\ell)\}_{k=1}^{\infty}$ is a complete orthogonal set in $L^2(0, \ell)$. Write your own script `fourierSineCoeffs.m` which takes exactly the same inputs as `fourierCoeffs.m`, and returns the appropriate coefficients so that, for a user-specified $f \in L^2(0, \ell)$,

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x/\ell)$$

holds (in the mean-square convergence sense) on the interval $[0, \ell]$. Then write a function `fsSine_approx.m` that plays a role analogous to `fs_approx.m`. Fix $\ell = 1$, choose $f(x) = 1$, and plot various truncated Fourier sine series approximations. Change the number of terms kept in the truncated series, as well as the size of the interval of x -values on which you plot f and its series together. (At least look at plots in $[-1, 1]$ and an interval that is twice as wide but contains $[-1, 1]$.) Repeat the process for $f(x) = x$. Just what function, exactly, is the truncated series converging to, in each case?

- (c) Repeat the process above, this time using the complete orthogonal set $\{1, \cos(k\pi x/\ell)\}_{k=1}^{\infty}$, writing scripts `fourierCosineCoeffs.m` and `fsCosine_approx.m` and testing them out with the same functions f .
2. A. Do Exercise 8.2.
- B. Use this result to explain why, if f were an odd function on $[-\ell, \ell]$, one would expect half of the coefficients returned by the script `fourierCoeffs.m` (in the previous problem) to be zero. Which half? What if f were an even function?
3. Do Exercise 8.19, part (a).
4. In this exercise we explore the properties of a complete orthonormal system on $(-\infty, \infty)$ called the Haar wavelets. Let ϕ be the function

$$\phi(x) = \begin{cases} 1, & \text{if } x \in [0, 1), \\ 0, & \text{otherwise.} \end{cases}$$

Let $\psi(x) = \phi(2x) - \phi(2x-1)$. The Haar wavelets are the (doubly-subscripted) functions

$$\psi_{mn}(x) := 2^{m/2} \psi(2^m x - n), \quad m, n = 0, \pm 1, \pm 2, \dots$$

- (a) Sketch graphs (you may do so by hand if you like) for ψ and ψ_{mn} , $m, n = 0, \pm 1, \pm 2$ (representatives from this range of m, n).
- (b) Prove that the ψ_{mn} are mutually orthogonal and that they are normalized (i.e., that $\|\psi_{mn}\|_{L^2(\mathbb{R})} = 1$ for each m, n). Note that the word *orthonormal* means *orthogonal + normal*.
- (c) Suppose $f \in L^2(\mathbb{R})$. The completeness of the Haar wavelets means that f has a generalized Fourier series expansion

$$f(x) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn} \psi_{mn}(x).$$

Find a formula (integral form, but simplified as much as possible) for the coefficients c_{mn} .

5. In class we considered a sequence of “hat”-shaped functions $\{g_n(\cdot)\}_{n=1}^{\infty}$ restricted to the domain $0 \leq x \leq 1$ that converges pointwise on $[0, 1]$ to the zero function. A formula for these functions is

$$g_n(x) := \begin{cases} 2^n x, & \text{if } 0 \leq x < 2^{-n}, \\ 2 - 2^n x, & \text{if } 2^{-n} \leq x < 2^{1-n}, \\ 0, & \text{if } 2^{1-n} \leq x \leq 1. \end{cases}$$

- (a) Show that $g_n \rightarrow 0$ (converges to the zero function) in the mean-square sense.
- (b) Find a sequence of functions $\{f_n(\cdot)\}_{n=1}^{\infty}$ that converges pointwise on $[0, 1]$ to the zero function, but does not converge to the zero function in the mean square. Do you think that all such examples of sequences $\{f_n(\cdot)\}_{n=1}^{\infty}$ must satisfy $\|f_n\|_{\infty} \rightarrow \infty$ as $n \rightarrow \infty$? If so, can you prove it?

Our hat-function example from class showed that it is possible for a sequence of functions to converge pointwise without converging uniformly. Part (b) indicates that pointwise convergence is possible without mean square convergence. Our example with the power functions $g_n(x) = x^n$ restricted to the interval $[0, 1]$ showed that mean square convergence is possible without pointwise convergence. However, Proposition 9.1 establishes that, at least on closed intervals $[a, b]$ of finite length, uniform convergence always implies the other two.