MATH 333: Partial Differential Equations

Problem Set 6, Final version

Due Date: Wed., Oct. 18, 2006

1. Consider the derivative operator $A = d^4/dx^4$ acting on some subset of functions ϕ in $L^2(0,1)$ which satisfy $\phi(0) = \phi'(0) = 0 = \phi(1) = \phi'(1)$. This operator arises in the Bernoulli-Euler model of transverse vibrations of beam clamped at both ends. Show that A is symmetric, positive definite.

2. Read through Exercise 2.16. Do parts (b) and (c).

3. Do Exercise 2.17.

4. Do Exercise 2.18.

5. Let $\alpha \in \mathbb{R}$ be fixed, and consider the linear differential operator given by

$$Lu \;:=\; -u'' + \alpha u,$$

   restricted to the domain $\mathcal{C}_0^2(0,1)$. By definition, for a certain (possibly nonreal) number λ to be an eigenvalue of a linear operator L, there needs to exist a function $u \in \mathcal{C}_0^2(0,1)$ not identically zero and satisfying the equation $Lu = \lambda u$.

   (a) Without actually finding any eigenvalues, show that an eigenvalue λ of this particular operator must be real and satisfy $\lambda > \alpha$.

   (b) Now do Exercise 2.24.

6. This exercise is a follow-up to problem 10(a) on Problem Set 1. Suppose V is an n-dimensional vector space (it need not, however, be $\mathbb{R}^n$) with *basis* $\{u_1, u_2, \ldots, u_n\}$. This means that the u_j are linearly independent

$$c_1 u_1 + \cdots + c_n u_n = 0 \qquad \text{implies} \qquad c_1 = \cdots = c_n = 0,$$

   and *span* V—i.e., every $v \in V$ can be written as a *linear combination* of the u_j or, to say it yet another way, for all $v \in V$, there exist scalars $c_1, \ldots, c_n$ (specific to the choice of v) such that

$$v \;=\; c_1 u_1 + c_2 u_2 + \cdots + c_n u_n.$$

   (a) Show that no $v \in V$ can be written as a linear combination of the u_j in two different ways. That is, show if

$$v \;=\; \sum_{j=1}^{n} c_j u_j \qquad \text{and} \qquad v = \sum_{j=1}^{n} d_j u_j,$$

   then $c_1 = d_1$, $c_2 = d_2$, $\ldots$, $c_n = d_n$.

(b) Assume now that V is an inner product space with inner product $\langle \cdot, \cdot \rangle$, and that the u_j are mutually orthogonal. Under these conditions, show that the scalars c_j are given by

$$c_j \;=\; \frac{\langle v, u_j \rangle}{\langle u_j, u_j \rangle}.$$