MATH 333: Partial Differential Equations

Problem Set 5, Final version

Due Date: Wed., Oct. 11, 2006

1. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$. Prove the following statement: An n-by-n matrix A is symmetric if and only if $\langle A\boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{x}, A\boldsymbol{y} \rangle$ for all $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$. Recall that, by definition, A is symmetric precisely when $A = A^T$.
   Hint: Use the fact that $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{y}^T\boldsymbol{x}$ where the right-hand side is matrix multiplication between the 1-by-n vector $\boldsymbol{y}^T$ and the n-by-1 vector $\boldsymbol{x}$. Also, recall that for a general matrix product, $(AB)^T = B^T A^T$. (Note: This is Exercise 2.21.)

2. Do Exercise 2.2. Use Examples 2.1 and 2.2 as a guide.

3. Do Exercise 2.12. When solving letter (c) of 2.2 numerically, take $a = 1$. Though you have not done Exercise 2.11, you can modify the following code, written in OCTAVE to produce numerical solutions to Example 2.5 (from p. 48) and to plot them (i.e., like Figures 2.2 and 2.3).

```
octave:1> function y = f(x)
> y = (3*x + x.^2) .* exp(x);
> end
octave:2> function A = buildA(n)
> A = diag(2*ones(n,1));
> B = diag((-1)*ones(n-1, 1));
> B = [zeros(1,n); B zeros(n-1, 1)];
> A = A + B + B';
> end
octave:3> function b = buildb(n)
> b = f([0:(1/(n+1)):1]')/(n+1)^2;
> b = b(2:n+1);
> end
octave:4> A = buildA(5);
octave:5> b = buildb(5);
octave:6> v = [0; A\b; 0];
octave:7> x = [0:0.01:1]';
octave:8> sparser_x = [0:(1/6):1]';
octave:9> function y = u(x)
> y = x.*(1 - x).*exp(x);
> end
octave:10> plot(x, u(x), 'b-', sparser_x, v, 'r-')
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   Determine numerical solutions for each of parts (a)-(c) using $n = 10$. Your submitted work should include the OCTAVE code that generates your solutions, the values of the solution vectors (vectors in $\mathbb{R}^{12}$), and plots of these solutions against the solutions of the corresponding continuous problems (the solutions found in the previous exercise).

As for the instructions concerning the error E_h, do this only for letter (a). In this case, carry out a numerical solution of (a) for various choices of n (like the choices of n presented in Table 2.1, p. 50) and determine E_h, the maximum difference (over all grid points) between the numerical solution and the solution of the continuous problem. What you hand in should include evidence that you are computing E_h correctly, along with a table akin to Table 2.1.

4. Do Exercise 2.14.