MATH 333: Partial Differential Equations

Problem Set 3, Final version

Due Date: Wed., Sept. 27, 2006

1. Do Exercise 1.10. Here are some suggestions. A similar heat problem, one with $\epsilon = 1$, is stated in equations (1.48), (1.49) of your text, and its solution appears in equation (1.55). You can use a change of variables—like the change from (x, t) to (ξ, η) on p. 15 in the context of the wave equation, or the scaling of the wave equation we did in class—to transform your problem into the one stated in (1.48), (1.49). I suggest doing something like

   $$\xi = ax, \qquad \eta = bt,$$

   for constants a, b which are as-yet undetermined, and work things out choosing a, b along the way so that the IVP (1.48), (1.49) pops out. Once you have done this, you will find the solution (1.55) useful for your problem as well.

   Now one advantage of using a mathematical software package like Octave is that it already has built-in capabilities for performing numerical integration. (There is no need for you to look up Simpson's rule in your calculus text and write a routine to implement it!) Octave uses quadrature methods to carry out most integrals, which explains the name of its built-in integrating function. If you want the value of

   $$\int_0^{\pi} \sin x \, dx,$$

   you can type

   ```
   > quad('sin', 0, 1)
   ans = 2
   ```

   Any built-in function (`tan(), asec(), exp(), log(),` etc.) of one real variable may be integrated in this way—just give the name of the function in quotes (I believe either single or double quotes work). (A word of caution: You should type

   > `help` *functionName*

   to see precisely what a function does before using it. For instance, you will find that there is no function `ln()`, while `log()` gives natural log values, not log values base 10.) You can even approximate improper integrals like $\int_{-\infty}^{0} e^x \, dx$ with commands like

   ```
   > quad('exp', -Inf, 0)
   ans = 1
   ```

   When the function you wish to integrate is not built-in, you must define it. For instance, to calculate

   $$\int_0^1 \frac{\sin x}{\sqrt{x}} \, dx,$$

   (also an improper integral) you can execute these commands:

```
> function y=f(x), y=sin(x)/sqrt(x); end
> quad('f', 0, 1)
ans = 0.62054
```

The "`function ...`" command above defines the integrand and names it f, essentially adding it temporarily (until you `clear f`, or `quit` the program) to the list of built-in commands. When you want a certain user-defined function to be more permanent (i.e., to be able to use the same function again in OCTAVE sometime in the future), the "`function ...`" line can be saved in an `.m` file called `f.m`. Do read Sections 6–8 in this .pdf file for more information about how to save and retrieve useful command sequences (i.e., computer programs) and functions (ones you have defined) for OCTAVE.

Now, the functions $u(x,t)$ you must plot for this exercise require that you carry out numerical integration for each fixed value of (x,t). While $t=1$ is fixed for the entire problem, the x-values for your plot are to roam through the interval $[-1,1]$. You will not be in position to plot $u(x,1)$ on this interval until you compute $u(x,1)$ for a representative list—say, a list like

```
> x = [-1:0.1:1]';
```

—of points $x \in [-1,1]$. You should find the following code helpful. It plots $\sin x$ on the domain $[0,\pi]$ in two different ways:

```
> x = [0:0.01:pi]';
> plot(x, sin(x), 'r-')   % plot sin(x) the easy way, pts spaced every 0.01
> hold on   % keeps 1st plot on graph when 2nd plot is added

> sparser_x = [0:0.3:pi];
> ys_via_integration = [];   % start out with empty vector
> for k = 1:length(sparser_x)
>   ys_via_integration = [ys_via_integration; quad('cos', 0, sparser_x(k))];
> end
> plot(sparser_x, ys_via_integration, 'b-')
>    % y-values of sin(x) have been obtained via numerical integration
>    % of cos(x); pts are spaced every 0.3 and connected via blue lines
>    % to make it easy to tell the 2nd plot from the 1st.
> hold off
```

2. (a) Show that, if $F(s) = \mathcal{L}(f(t))$, then $\mathcal{L}(H(t-a)f(t-a)) = e^{-as}F(s)$, where H is the Heavyside unit step function of (1.49), p. 18.

   (b) Use the above result to show that the IBVP

$$\begin{aligned} u_{tt} &= c^2 u_{xx}, && x>0,\ t>0, \\ u(0,t) &= g(t), && t>0, \\ u(x,0) &= u_t(x,0) = 0, && x>0. \end{aligned}$$

   is $u(x,t) = H(t - x/c)g(t - x/c)$.

3. Consider the first-order initial boundary value problem with $c > 0$ (a constant):

$$\begin{aligned} \frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} &= 0, \quad x > 0, \quad t > 0, \\ w(0,t) &= f(t), \qquad t > 0, \\ w(x,0) &= 0, \qquad\quad x \geq 0. \end{aligned}$$

   Solve this problem in two ways:

   (i) Use the method of Laplace transforms.

   (ii) Use the method of characteristics.

   (Naturally, if you do not get the same answer using both methods, something is not quite right.)

4. Do Exercise 1.11.

5. Do Exercise 1.17, part (a). Instead of the formula for S provided in your text, assume $k > 0$ (constant) and take S to be

$$S(x,t) := \frac{1}{\sqrt{4\pi k t}}\, e^{-x^2/(4kt)}.$$

6. Do Exercise 12.1(a).

7. Do Exercise 12.3.

8. Do Exercise 12.4.