MATH 333: Partial Differential Equations

Problem Set 2, Final version

Due Date: Wed., Sept. 20, 2006

1. Do as much of Exercise 1.5 as you have time for, but hand in parts (b) and (c).
2. Do Exercise 1.6.
3. Consider the Cauchy problem for the wave equation as stated in (1.59), Exercise 1.8.
   (a) Verify that the formula for u in (1.59a) is, indeed, a solution of (1.59). (This serves as a check on the derivation of the solution. Alternatively, in situations where a solution has been guessed, this kind of calculation serves as a test of the correctness of the guess.) To differentiate the integral part of (1.59a), you will need to employ the Fundamental Theorem of Calculus, Part I and the chain rule. As an example, remember that

   $$\frac{d}{dx}\int_{g(x)}^{f(x)} w(\xi)\,d\xi = \frac{d}{dx}\int_0^{f(x)} w(\xi)\,d\xi - \frac{d}{dx}\int_0^{g(x)} w(\xi)\,d\xi, \qquad \text{and}$$

   $$\begin{aligned}\frac{d}{dx}\int_0^{f(x)} w(\xi)\,d\xi &= \frac{d}{dx}\int_0^{u} w(\xi)\,d\xi \qquad (\text{where } u = f(x))\\ &= \left(\frac{d}{du}\int_0^{u} w(\xi)\,d\xi\right)\frac{du}{dx} \qquad (\text{by the chain rule})\\ &= w(u)\frac{du}{dx} \qquad (\text{by FTC, Part I})\\ &= w(f(x))f'(x).\end{aligned}$$

   You should find that the solution formula (1.59a) contains an error. Either guess the way to correct it, or do a quick internet search on the keywords "D'Alembert's solution" to find the corrected one.

   (b) Write out the solution of Exercise 1.9 employing the corrected formula (1.59a). Evaluate the definite integrals so that no integrals remain in your final expression for u.
4. In class we considered problem (1.59) and, letting $v(x,t) = u_t(x,t) + cu_x(x,t)$, we arrived at

   $$v(x,t) = \psi(x+ct) + c\phi'(x+ct).$$

   Complete the derivation of (1.59a) (corrected form) by using the method of characteristics on the remaining 1st-order Cauchy problem

   $$\begin{aligned} u_t + cu_x = v, &\quad x \in \mathbb{R}, t > 0,\\ u(x,0) = \phi(x), &\quad x \in \mathbb{R}.\end{aligned}$$

   (Note that this is more like Exercise 1.14 than it is Exercise 1.8; it may be worthwhile to read through that exercise.)

5. Suppose we wish to solve the 1st-order linear nonhomogeneous Cauchy problem

$$\begin{array}{ll} u_t + a(x,t)u_x + b(x,t)u = f(x,t), & x \in \mathbb{R},\ t > 0 \\ u(x,0) = \phi(x), & x \in \mathbb{R}. \end{array} \tag{1}$$

Consider the related problems

$$\begin{array}{ll} v_t + a(x,t)v_x + b(x,t)v = 0, & x \in \mathbb{R},\ t > 0 \\ v(x,0) = \phi(x), & x \in \mathbb{R}, \end{array} \tag{2}$$

and

$$\begin{array}{ll} w_t + a(x,t)w_x + b(x,t)w = f(x,t), & x \in \mathbb{R},\ t > 0 \\ w(x,0) = 0, & x \in \mathbb{R}. \end{array} \tag{3}$$

Without attempting to solve problems (2), (3) (i.e., without expressions for their solutions v and w), show that when v and w solve (2), (3) respectively, then $u = v + w$ solves (1). (Don't forget to verify that u satisfies the appropriate initial condition.)

6. Consider the source-free, 1-dimensional fundamental conservation law $\theta_t + \phi_x = 0$, where $\theta(x,t)$ is the density (quantity/length) of some substance in a 1-dimensional "tube" and ϕ is the flux (quantity/time). In particular, suppose that θ is the energy density in a 1-dimensional metal rod. Let us assume that energy density is proportional to temperature $u(x,t)$:

$$\theta(x,t) = \rho C u(x,t),$$

where ρ is the mass density (mass/length) and C is the specific heat of the metal. One might expect that energy flows down the gradient; one way to model this would be to write $\phi = -Ku_x$, where K is called *thermal conductivity*. Assuming ρ, C and K are constant, derive the one-dimensional heat equation

$$u_t \;=\; ku_{xx}, \qquad \text{where} \quad k \;:=\; \frac{K}{\rho C}.$$

How does this equation change if we presume K to depend upon temperature?

7. Let u satisfy the 1-dimensional initial boundary value problem (IBVP) for the heat equation

$$\begin{array}{ll} u_t \;=\; ku_{xx}, & 0 < x < \ell,\ \ t > 0, \\ u(0,t) \;=\; u(\ell,t) \;=\; 0, & t > 0, \\ u(x,0) \;=\; \phi(x), & 0 \le x \le \ell. \end{array}$$

Show that, for all $t \ge 0$,

$$\int_0^\ell u^2(x,t)\,dx \;\le\; \int_0^\ell \phi^2(x)\,dx.$$

(Hint: Define

$$E(t) \;:=\; \int_0^\ell u^2(x,t)\,dx,$$

and assume that time differentiation can pass through the integral to get that $E'(t) \leq 0$ for all $t > 0$. Note: The fact that u satisfies the IBVP is instrumental in this process.) In what sense is $E(\cdot)$ a measurement of the temperature of the bar? What does the inequality you are to prove say about the stability of solutions to this heat problem? Explain.

8. Take 30 minutes to browse through the chapter on Laplace transforms in your textbook on ordinary differential equations. In particular, re-familiarize yourself with the definition of the Laplace transform of f (in my copy of Boyce & DiPrima this comes about one-fourth of the way down on the 4th page of Ch. 6), look at some examples of how Laplace transforms are computed (Section 6.1 in B&D), look at some examples of how the Laplace transform is employed in solving IVPs (for B&D, those in Section 6.2 probably suffice), and aquaint (or re-aquaint) yourself with the *convolution theorem* (Thm. 6.6.1, I believe, in B&D). Another item (likely in that chapter) to recall is the *unit impulse function* (a.k.a. the Dirac *delta function*); remind yourself how it was introduced as the limit of functions

$$d_\tau(t) = \begin{cases} 1/(2\tau), & -\tau < t < \tau, \\ 0, & \text{otherwise,} \end{cases}$$

as $\tau \to 0$.